Semidefinite programming in quantum information theory

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Outline

Basic ideas

- A pure state of N qubits can be described by state vector $|\Psi\rangle$ (=column vector) of 2^N complex elements fulfilling

$$\langle\Psi|\Psi\rangle = 1.$$

- A mixed state is some mixture of pure states

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|,$$

  where p_k are probabilities.

- For N qubits, it is of size $2^N \times 2^N$.

Outline

Basic ideas II

- In quantum physics, the density matrix ϱ is a positive semidefinite matrix
$$\varrho \geq 0.$$
- Its trace is one
$$\mathrm{Tr}(\varrho) = 1$$
and it is Hermitian
$$\varrho = \varrho^\dagger.$$
These conditions can easily be included in a semidefinite program.
- When we measure an operator X, the expectation value is
$$\langle X \rangle = \mathrm{Tr}(\varrho X).$$

Basic ideas III

- Let us see a simple example. We look for the minimum of

$$\langle X \rangle = \mathrm{Tr}(\varrho X)$$

  with the condition

$$\langle Y_n \rangle = \mathrm{Tr}(\varrho Y_n) = y_n$$

  for $n = 1, 2, .., N$, where X, Y_n are operators.

- We optimize over ϱ density matrices.

- This is again doable with semidefinite programming, although, there are better ways to do it.

N representability problem IV

- Find ϱ of N qudits such that for the reduced states we have

$$\mathrm{Tr}_{I\backslash\{m,n\}}(\varrho) = \varrho_{mn},$$

  where $I = \{1, 2, ..., N\}$.

  A. J. Coleman, Rev. Mod. Phys. 35, 668 (1963),

  for a summary of the literature see in Doherty, Parillo, Spedalieri, PRA 2005.

- Note that

$$\mathrm{Tr}_{I\backslash\{m,n\}}(\varrho)$$

  is a matrix with elements that depend linearly on the elements of ϱ.

Basic ideas V

- Let us see a simple example. We look for the minimum of

$$\langle X_1 \rangle^2 + \langle X_2 \rangle^2 = \mathrm{Tr}(\varrho X_1)^2 + \mathrm{Tr}(\varrho X_2)^2$$

with the condition

$$\langle Y_n \rangle = \mathrm{Tr}(\varrho Y_n) = y_n$$

for $n = 1, 2, .., N$.

- We optimize over ϱ density matrices.

- This is again doable with semidefinite programming, minimising $t_1 + t_2$ using the constraints

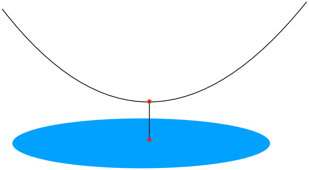
$$\begin{pmatrix} t_k & \mathrm{Tr}(\varrho X_k) \\ \mathrm{Tr}(\varrho X_k) & 1 \end{pmatrix} \geq 0$$

for $k = 1, 2$.

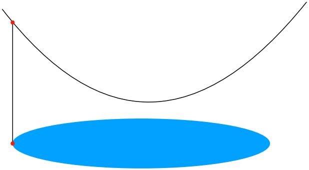
Outline

Solvable vs. not solvable by SDP

- Thus, we can minimize a convex function over the convex set of density matrices.



- However, we cannot maximize a function over the convex set of density matrices efficiently - the maximum is taken at the boundaries.



Outline

Local Operation and Classical Communications

Definition

Local Operation and Classical Communications (LOCC):

- Single-party unitaries,
- Single party von Neumann measurements,
- Single party POVM measurements,
- We are even allowed to carry out measurement on party A and depending on the result, perform some other operation on party B ("Classical Communication").

Pure states: product states vs. entangled states

- A pure two-qubit state is either a product state

$$|\Psi_1\rangle_A \otimes |\Psi_2\rangle_B,$$

  or an entangled state.

- From a single copy of any pure entangled two-qubit state, we can get to any other entangled two-qubit state through Stochastic Local Operations and Classical Communication (SLOCC).

  That is, for any entangled $|\Psi\rangle$ and $|\Phi\rangle$, there are invertible X and Y such that

$$|\Psi\rangle = X \otimes Y|\Phi\rangle.$$

  Note that X and Y do not have to be Hermitian.

- This is not true for higher dimensional systems.

Pure states: product states vs. entangled states

- Examples for separable states

$$|0\rangle \otimes |0\rangle, \quad |1\rangle \otimes |1\rangle.$$

- An example for entangled states

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle).$$

Mixed states: separable states vs. entangled states

- For the mixed case, the definition of a separable state is (Werner 1989)

$$\rho_{\text{sep}} = \sum_k p_k [\rho_k^{(1)}]_A \otimes [\rho_k^{(2)}]_B.$$

  A state that is not separable, is entangled.

- It is not possible to create entangled states from separable states, with LOCC.

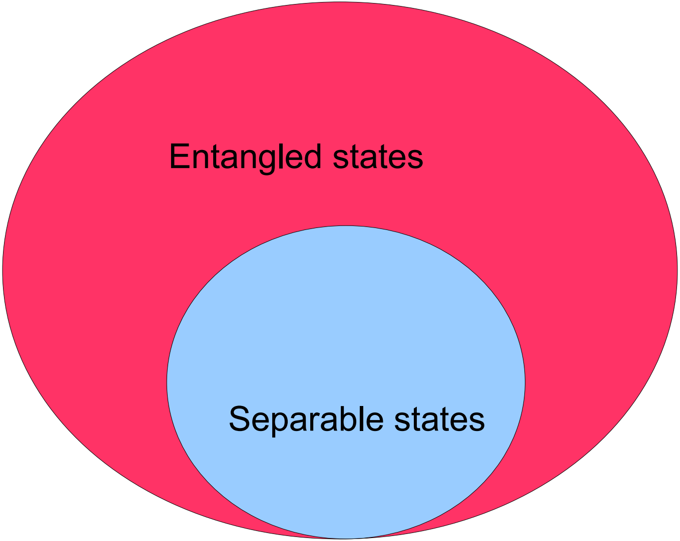
- From many copies of two-qubit mixed entangled states, we can always distill a singlet using Local Operations and Classical Communication (LOCC).

- This is not true for higher dimensional systems. Not all quantum states are distillable.

Distillation

- From many entangled particle pairs we want to create fewer strongly entangled pairs with LOCC.

Convex sets



Bipartite systems I

- Naive question: can we decide whether a state is separable with SDP? No, because we would need a constraint of the type

$$\varrho = (\varrho_1)_A \otimes (\varrho_2)_B.$$

- Alternatively, we would need a constraint for the reduced states of the *nth* subsytem

$$\mathrm{Tr}(\varrho^2_{\mathrm{red},n}) = 1.$$

Bipartite systems II

- How can we check separability using a brute force method? We can look for $\rho_k^{(1)}, \rho_k^{(2)}$ numerically.
- Simpler problem, maximum for an operator expectation value for separable states

$$\max_{\rho_{\text{sep}}} \text{Tr}(X\rho_{\text{sep}}) = \max_{\Psi_1,\Psi_2} \langle\Psi_1|\langle\Psi_2|X|\Psi_2\rangle\Psi_1\rangle.$$

- Numerically, we can try to find the maximum. In practice, we will find the maximum or something lower.

Outline

The positivity of the partial transpose (PPT) criterion

Definition

For a separable state ϱ living in AB, the partial transpose is always positive semidefinite

$$\varrho^{TA} \geq 0.$$

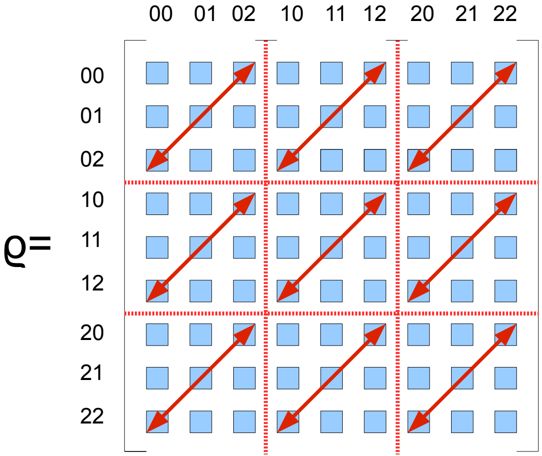
If a state does not have a positive semidefinite partial transpose, then it is entangled. A. Peres, PRL 1996; Horodecki *et al.*, PLA 1997.

- Partial transpose means transposing according to one of the two subsystems.
- For separable states

$$(T \otimes \mathbb{1})\varrho = \varrho^{TA} = \sum_k p_k (\varrho_k^{(1)})^T \otimes \varrho_k^{(2)} \geq 0.$$

The positivity of the partial transpose (PPT) criterion II

- How to obtain the partial transpose of a general density matrix? Example: 3×3 case.



The positivity of the partial transpose (PPT) criterion III

- If

$$\varrho^{TA} \geq 0$$

is violated then the state is entangled!

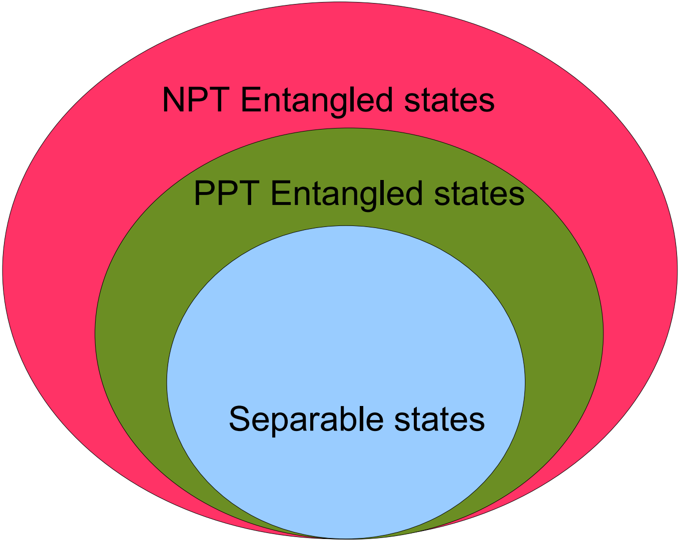
- For 2×2 and 2×3 systems it detects all entangled states.

- For larger systems, there are entangled states for which

$$\varrho^{TA} \geq 0.$$

hold. They are bound entangled, not distillable.

Convex sets


NPT Entangled states
PPT Entangled states
Separable states

The positivity of the partial transpose (PPT) criterion IV

- Semidefinite programming can be used to optimize over PPT states.
- Find the maximum of an operator expectation value for PPT states:

Maximize

$$\langle X \rangle_\varrho \equiv \mathrm{Tr}(X\varrho)$$

such that

$$\begin{aligned} \varrho &= \varrho^\dagger, \\ \varrho &\geq 0, \\ \varrho^{TA} &\geq 0, \\ \mathrm{Tr}(\varrho) &= 1. \end{aligned}$$

The positivity of the partial transpose (PPT) criterion V

- This is like finding an upper bound on the maximum for separable states.
- In practice, we often find the maximum for separable states.

G. Tóth, W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, and H. Weinfurter, New J. Phys. 2009.

The positivity of the partial transpose (PPT) criterion V

- We can ask: is there a PPT fulfilling certain constraints?

  Look for ϱ such that

$$
\begin{aligned}
\varrho &= \varrho^{\dagger}, \\
\varrho &\geq 0, \\
\varrho^{TA} &\geq 0, \\
\mathrm{Tr}(\varrho) &= 1, \\
\mathrm{Tr}(X_k \varrho) &= x_k \text{ for } k = 1, 2, .., K.
\end{aligned}
$$

- If there is not such a ϱ then the state fulfilling the constraints is not PPT, and it is entangled (or it is not physical).
- One can use this to detect entanglement in experiments.

Measuring entanglement, bipartite case

- Entanglement of formation:
  - Pure states: Von Neumann entropy of the reduced state

$$E_F(\varrho) = S[\mathrm{Tr}_A(\varrho)],$$

where

$$S(\varrho) = -\mathrm{Tr}(\varrho \log \varrho).$$

  - Mixed states: Defined by a convex roof construction

$$E_F(\varrho) = \min_{\{|\Psi_k\rangle, p_k\}: \varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|} \sum_k p_k E_F(|\Psi_k\rangle).$$

- Negativity: = (-1) times the sum of the negative eigenvalues of the partial transpose. (Vidal, Werner)

Measuring entanglement, bipartite case II

- Entanglement of formation measures the number of singlets needed to create the state with LOCC.
- For separable states it is zero.
- For the singlet

$$(|01\rangle - |01\rangle)\,\sqrt{2},$$

or

$$(|00\rangle + |11\rangle)\,\sqrt{2},$$

it is 1.

Outline

Three-qubit mixed states

Six classes:

Class #1: fully separable states $\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes \varrho_3^{(k)}$

Class #2: (1)(23) biseparable states $\sum_k p_k \varrho_1^{(k)} \otimes \varrho_{23}^{(k)}$, not in Class #1

Class #3: (12)(3) biseparable states $\sum_k p_k \varrho_{12}^{(k)} \otimes \varrho_3^{(k)}$, not in Class #1

Class #4: (13)(2) biseparable states $\sum_k p_k \varrho_{13}^{(k)} \otimes \varrho_2^{(k)}$, not in Class #1

Class #5: W-class states:
mxtr of pure (W ∪ Bisep ∪ Sep)-class states, not in Classes #1-4

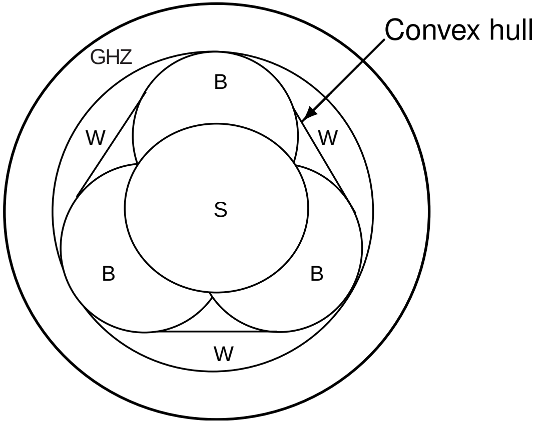
Class #6: GHZ-class states: mxtr of pure (GHZ ∪ W ∪ Bisep ∪ Sep)-class states, not in Classes #1-5

Biseparable states: mixture of states of classes #2, #3 and #4.

Three-qubit mixed states II

- The extension of the classification of pure states to mixed states leads to convex sets:

A. Acín, D. Bruss, M. Lewenstein, A. Sanpera, Phys. Rev. Lett. 87, 040401 (2001)



Witnesses for GHZ and W-class states

Entanglement witnesses for detecting states of a given class:

GHZ-class states

$$\mathcal{W}_{\mathrm{GHZ}}^{(P)} := \frac{3}{4}\mathbb{1} - |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|.$$

W-class states

$$\mathcal{W}_{\mathrm{W}}^{(P)} := \frac{2}{3}\mathbb{1} - |\mathrm{W}\rangle\langle\mathrm{W}|.$$

$$\mathcal{W}_{\mathrm{GHZ}}^{(P)} := \frac{1}{2}\mathbb{1} - |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|.$$

$\mathrm{Tr}(\mathcal{W}\varrho) < 0$ signals entanglement of the given type.

A. Acín, D. Bruss, M. Lewenstein, A. Sanpera, Phys. Rev. Lett. 87, 040401 (2001)

Outline

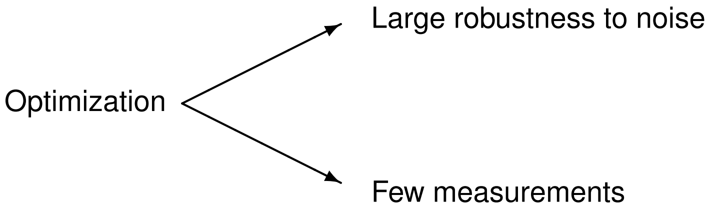
Aims when designing a witness

Definition

An entanglement witness $\mathcal{W}$ is an operator that is positive on all separable (biseparable) states.

Thus, $\mathrm{Tr}(\mathcal{W}\varrho) < 0$ signals entanglement (genuine multipartite entanglement). Horodecki 1996; Terhal 2000; Lewenstein, Kraus, Cirac, Horodecki 2002

There are two main goals when searching for entanglement witnesses:



Simple idea for witnesses

- A witness can be defined for a bipartite systems as

$$\mathcal{W} = c\mathbb{1} - M,$$

where

$$c = \max_{|\Psi_1\rangle \otimes |\Psi_2\rangle} \langle M \rangle \, .$$

Find a lower bound on the maximum of the expectation value for separable states

- Maximize numerically

$$\mathrm{Tr}(X|\Psi_1\rangle\langle\Psi_1|\otimes|\Psi_2\rangle\langle\Psi_2|)$$

over

$$|\Psi_1\rangle\langle\Psi_1|, |\Psi_2\rangle\langle\Psi_2|.$$

- We can get a lower bound on the maximum. (We might not find the maximum.)

Find an upper bound on the maximum of the expectation value for separable states

- Maximize

$$\mathrm{Tr}(X\varrho_{AB})$$

over ϱ_{AB} fulfilling

$$\begin{aligned} \varrho_{AB} &= \varrho_{AB}^{\dagger}, \\ \varrho_{AB} &\geq 0, \\ \varrho_{AB}^{TA} &\geq 0, \\ \mathrm{Tr}(\varrho_{AB}) &= 1. \end{aligned}$$

- We can get an upper bound on the maximum. (We might not find the maximum.)

Another simple idea for witnesses

- If $\mathcal{W}^{(P)}$ is a witness then $\mathcal{W}$ is also a witness if

$$\mathcal{W} - \alpha\mathcal{W}^{(P)} \geq 0$$

  for some $\alpha > 0$.

Optimizing witnesses

- Noisy state

$$\varrho(p_{\text{noise}}) = (1 - p_{\text{noise}})\varrho + p_{\text{noise}}\varrho_{\text{noise}}.$$

- The state is detected by a witness $\mathcal{W}$ if $\text{Tr}(\mathcal{W}\varrho) < 0$, which is the case if

$$p_{\text{noise}} < \frac{\text{Tr}(\mathcal{W}\varrho)}{\text{Tr}(\mathcal{W}\varrho) - \text{Tr}(\mathcal{W}\varrho_{\text{noise}})} =: p_{\text{limit}}.$$

- Let us assume that the witness is a linear combination of basis operators

$$\mathcal{W} = \sum_k c_k B_k.$$

- We look for the c_k such that p_{limit} is maximal and

$$\mathcal{W} - \alpha\mathcal{W}^{(P)} \geq 0$$

for some $\alpha > 0$.

Optimizing witnesses

1. Semi-definite programming can be used to find the witness $\mathcal{W}$ with the largest noise tolerance as explained in the beginning of section 3.1.1. The corresponding task can be formulated as

$$\begin{aligned} \textbf{minimize} \quad & \sum_k c_k \mathrm{Tr}(B_k \varrho_{\mathrm{noise}}), \\ \textbf{subject to} \quad & \sum_k c_k \mathrm{Tr}(B_k \varrho) = -1, \\ & \sum_k c_k B_k - \alpha \mathcal{W}^{(P)} \geqslant 0, \\ & \alpha > 0. \end{aligned} \tag{A.1}$$

Here ϱ is the state around which we detect entanglement. ϱ_{noise} is the noise, not necessarily white. The optimization is over α and the c_k's.

G. Tóth, W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, and H. Weinfurter, Practical methods for witnessing genuine multi-qubit entanglement in the vicinity of symmetric states, New J. Phys. 11, 083002 (2009).

Optimizing witnesses II

3.1.2. Three-setting witness. Similarly we can look for the optimal witness for the three-setting case. The result is

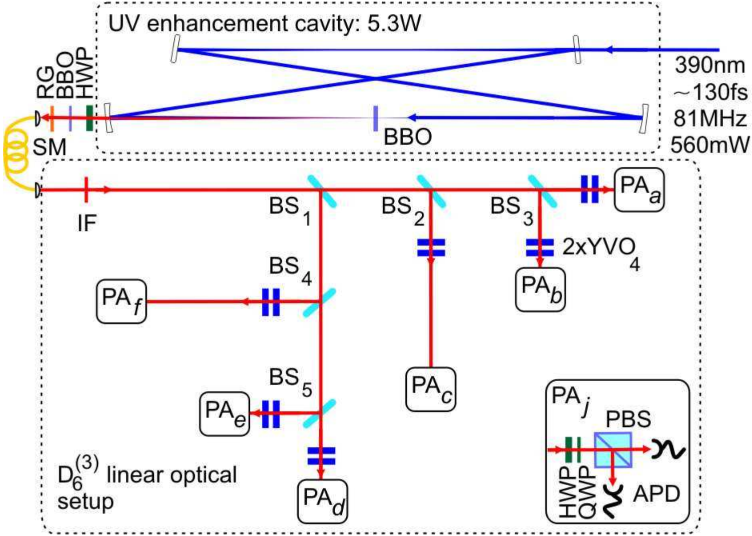
$$\mathcal{W}_{\mathrm{D}(6,3)}^{(P3)} := 1.5 \cdot \mathbb{1} - \tfrac{1}{45}(J_x^2 + J_y^2) + \tfrac{1}{36}(J_x^4 + J_y^4) - \tfrac{1}{180}(J_x^6 + J_y^6) + \tfrac{1007}{360} J_z^2 - \tfrac{31}{36} J_z^4 + \tfrac{23}{360} J_z^6. \qquad (29)$$

White noise is tolerated if $p_{\mathrm{noise}} < 0.2735$. It is easy to check that $\mathcal{W}$ is a witness as $\mathcal{W}_{\mathrm{D}(6,3)}^{(P3)} - 2.5\mathcal{W}^{(P)} \geqslant 0$.

G. Tóth, W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, and H. Weinfurter, Practical methods for witnessing genuine multi-qubit entanglement in the vicinity of symmetric states, New J. Phys. 11, 083002 (2009);

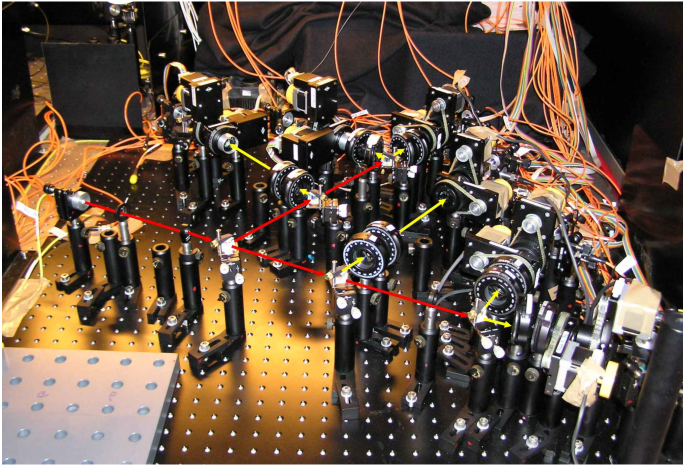
W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Tóth, and H. Weinfurter, Experimental entanglement of a six-photon symmetric Dicke state, Phys. Rev. Lett. 103, 020504 (2009).

An experiment: Dicke state with photons



An experiment: Dicke state with photons II

A photo of a real experiment (six-photon Dicke state, Weinfurter group, 2009):



Outline

Locally symmetric extensions

Definition (Locally symmetric extensions)

- Let us assume that ϱ_{AB} is a bipartite quantum state. Then, $\varrho_{ABB'}$ is a symmetric extension of ϱ_{AB} for the party B if

$$\mathrm{Tr}_{B'}\varrho_{ABB'} = \varrho_{AB}$$

and

$$\mathcal{P}_{BB'}\varrho_{ABB'}\mathcal{P}_{BB'} = \varrho_{ABB'},$$

$\mathcal{P}_{BB'}$ is the operator swapping B and B'.

- We can talk about a locally symmetric extension $\varrho_{ABB'B''}$ in an analogous way.

A. Doherty, P. A. Parillo, F. M. Spedalieri, Phys. Rev. Lett. 2002; Phys. Rev. A 2004; Phys. Rev. A 2005.

PPT locally symmetric extensions

Definition (PPT locally symmetric extensions)

- Let us assume that ϱ_{AB} is a bipartite quantum state. Then, $\varrho_{ABB'}$ is a PPT symmetric extension of ϱ_{AB} for the party B if

$$\mathrm{Tr}_{B'}\varrho_{ABB'} = \varrho_{AB},$$

$$\mathcal{P}_{BB'}\varrho_{ABB'}\mathcal{P}_{BB'} = \varrho_{ABB'},$$

  and

$$\varrho_{ABB'}$$

  is PPT with respect to all bipartitions.

- We can talk about a PPT locally symmetric extension $\varrho_{ABB'B''}$ and $\varrho_{ABB'B''B'''}$ in an analogous way.

- We call them 1 : 2 and 1 : 3 locally symmetric extensions.

Outline

Locally symmetric extensions for separable states

- Separable states have a PPT symmetric extension to arbitrary number of parties.

- For instance an $AB \to ABB'$ extension can be given as

$$\rho_{\mathrm{sep}} = \sum_k p_k [\rho_k^{(1)}]_A \otimes [\rho_k^{(2)}]_B. \to$$

$$\rho_{\mathrm{extension}} = \sum_k p_k [\rho_k^{(1)}]_A \otimes [\rho_k^{(2)}]_B \otimes [\rho_k^{(2)}]_{B'}$$

- It can be shown that entangled states do not have extensions to arbitrary many parties.

Algorithm for entanglement detection

- Find the $n_A : n_B$ PPT locally symmetric extension.
- If it does not exist then the state is entangled.
- If it exists then we have to try larger n_A and/or larger n_B. (In principle, we can restrict our attention to $n_A = 1$.)

Bosonic symmetry

- As a bosonic state, the extension can be efficiently stored even for many qubits.
- An N-qubit symmetric state can be stored in a $(N+1)\times(N+1)$ density matrix.

G. Tóth and O. Gühne, Entanglement and permutational symmetry, Phys. Rev. Lett. 102, 170503 (2009);

M. Navascues, M. Owari, M, B. Plenio, The power of symmetric extensions for entanglement detection, Phys. Rev. A. (2009).

Outline

Entanglement witnesses

- The dual problem gives an entanglement witness, if the state does not have an extension.
- With the witness the state is detected as entangled.

Outline

Entanglement

The entanglement of a bipartite quantum state

For pure states living on AB, it is defined as

$$E(|\Psi\rangle) = S[\mathrm{Tr}_A(|\Psi\rangle)],$$

for pure states, where S is an entropy.

For mixed states, it is defined with a convex roof as

$$E(\varrho) = \min_{\{p_k, |\Psi_k\rangle\}} \left(\sum_k p_k E(|\Psi_k\rangle) \right),$$

where $\{p_k, |\Psi_k\rangle\}$ is a decomposition to pure states

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

Linear entropy for pure states

- Linear entropy
$$S_{\rm lin}(\varrho) = 1 - {\rm Tr}(\varrho^2).$$

- Known: linear entropy of entanglement for pure states can be defined as an expectation value on two copies (AB and $A'B'$) as
$$E_{\rm lin}(|\Psi\rangle) = {\rm Tr}[\mathcal{A}_{\rm AA'} \otimes \mathbb{1}_{BB'} (|\Psi\rangle\langle\Psi|)_{AB} \otimes (|\Psi\rangle\langle\Psi|)_{A'B'}],$$
where
$$\mathcal{A}_{\rm AA'} := (\mathbb{1} - \mathcal{F})_{\rm AA'}$$
and $\mathcal{F}$ is the flip operator.

Linear entropy for mixed states: convex roof

- For mixed states

$$
\begin{aligned}
E_{\rm lin}(\varrho) &= \min_{\{p_k,|\Psi_k\rangle\}} \sum_k p_k E_{\rm lin}(|\Psi_k\rangle) = \\
&= \min_{\{p_k,|\Psi_k\rangle\}} \sum_k p_k {\rm Tr}(\mathcal{A}_{AA'}|\Psi_k\rangle\langle\Psi_k|^{\otimes 2}) \\
&= \min_{\omega_{12}} {\rm Tr}(\mathcal{A}_{AA'}\omega_{12}),
\end{aligned}
$$

  where ω_{12} are symmetric separable states, i.e.,

$$
\omega_{12} = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \otimes |\Psi_k\rangle\langle\Psi_k|.
$$

- **<u>This is the key step in our approach.</u>**

Surprise 1

- Mapping of the problem

Optimization over decompositions $\longrightarrow$ Optimization over symmetric separable states

- We connected the separability theory to a general mathematical problem.

How to calculate it

- The convex roof of the linear entropy can be written as

$$\begin{aligned} E_{\text{lin}}(\varrho) = \min_{\omega_{12}} \quad & \text{Tr}(\mathcal{A}_{AA'}\omega_{12}), \\ \text{s.t.} \quad & \omega_{12} \text{ is symmetric, separable,} \\ & \omega_1 = \varrho, \end{aligned}$$

where $\omega_1 \equiv \text{Tr}_2(\omega_{12})$.

- A lower bound can be obtained as with the PPT condition

$$\begin{aligned} E_{\text{lin}}(\varrho) = \min_{\omega_{12}} \quad & \text{Tr}(\mathcal{A}_{AA'}\omega_{12}), \\ \text{s.t.} \quad & \omega_{12} \text{ is symmetric PPT,} \\ & \omega_1 = \varrho, \end{aligned}$$

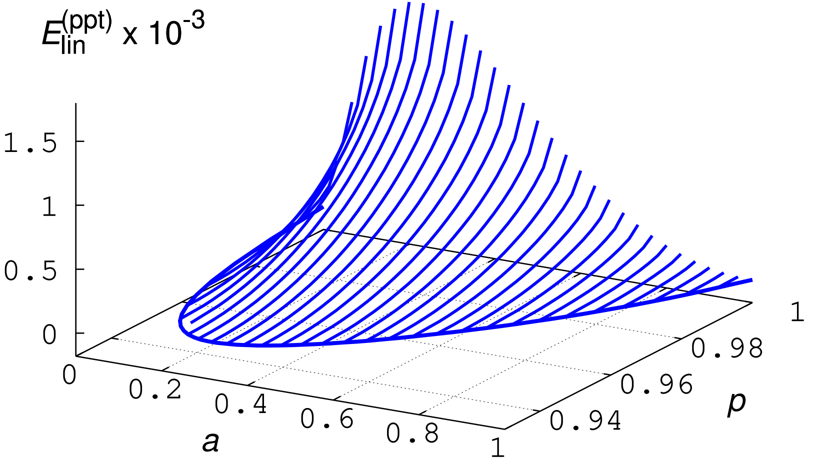
where $\omega_1 \equiv \text{Tr}_2(\omega_{12})$. This is a semidefinite program.

Surprise 2

- The lower bound
  - is nonzero for all states with a non-positive semidefinite partial transpose (NPPT).
  - is nonzero for some states with a positive semidefinite partial transpose (PPT).
- For all non-PPT states and for all states that do not have a $2:2$ symmetric extension we have a nonzero bound.
- Moreover, for all states having a 2:2 PPT symmetric extension the bound is zero.
  [Extensions: Doherty, Parrilo, Spedalieri, PRA 69, 022308 (2004)]

Example: Entanglement of a PPT state

- 3×3 Horodecki state mixed with white noise.
- a =parameter of the state, $1 - p$ =noise fraction



Outline

Wootters' Tangle

- The well-known tangle for three-qubits can be defined as a fourth-order polynomial in expectation values.

  A. Osterloh and J. Siewert, Phys. Rev. A 86, 042302 (2012).

- Hence, it can be obtained as an optimization over four-partite symmetric separable states

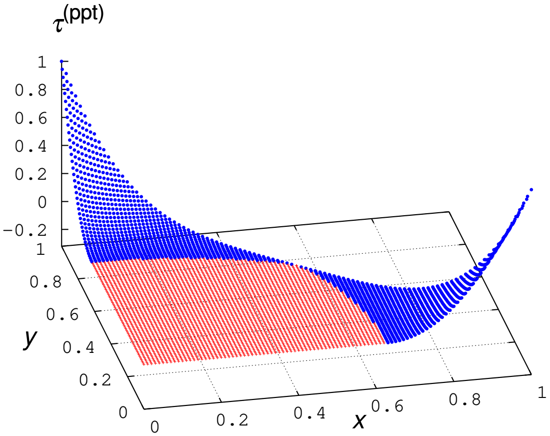
$$\begin{aligned} \tau(\varrho) = \min_{\omega_{1234}} \quad & \mathrm{Tr}(T\omega_{1234}), \\ \text{s.t.} \quad & \omega_{1234} \text{ symmetric, fully separable}, \\ & \omega_1 = \varrho, \end{aligned}$$

  where T is an operator (4 parties with 3 qubits each).

- Similar idea works: replace separable states by PPT states.

Example: tangle of a two-parameter family of states

$$\varrho(x,y) = x|\mathrm{GHZ}^+\rangle\langle\mathrm{GHZ}^+| + y|\mathrm{GHZ}^-\rangle\langle\mathrm{GHZ}^-| + (1-x-y)|\mathrm{W}\rangle\langle\mathrm{W}|$$



Outline

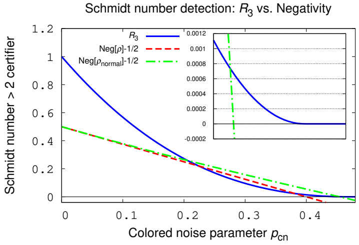
Other quantities

- Schmidt number. I.e., the convex roof of $R_3(|\Psi\rangle) = \sum_{i<j<k} \lambda_i \lambda_j \lambda_k$ tells us whether the Schmidt number is larger than 2.
- Entanglement vs. CHSH violation
- Lower bound on entanglement based on some measurement results
- Concave roof instead of convex roofs: E. of assistance
- Lower bound on quantum Fisher information based some measurement results.

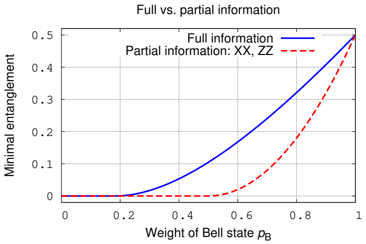
  [Tóth and Petz, PRA 2013.]
- One can get even a witness!!

  [For references, please G. Tóth, T. Moroder, and O. Gühne, Phys. Rev. Lett. (2015).]

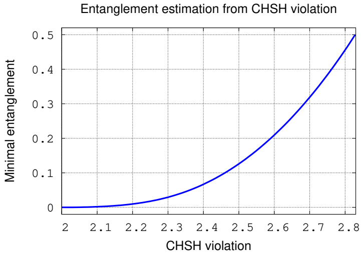
Examples



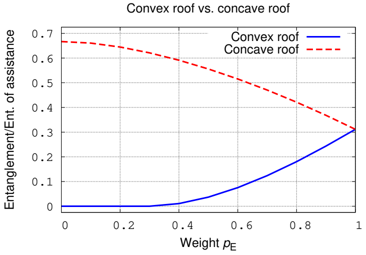
Schmidt number detection: R_3 vs. Negativity
R_3
Neg[ρ]-1/2
Neg[ρ_{normal}]-1/2
Schmidt number > 2 certifier
Colored noise parameter p_{cn}



Full vs. partial information
Full information
Partial information: XX, ZZ
Minimal entanglement
Weight of Bell state p_B



Entanglement estimation from CHSH violation
Minimal entanglement
CHSH violation



Convex roof vs. concave roof
Convex roof
Concave roof
Entanglement/Ent. of assistance
Weight p_E

Outline

A series of tighter and tighter lower bounds

- To strengthen the bound, a criterion stronger than PPT must be employed.
- For example, the method of PPT locally symmetric extensions can be used.

  [Doherty, Parrilo, Spedalieri, Phys. Rev. A 69, 022308 (2004)]
- Sequence of lower bounds $E_{\mathrm{lin}}^{(n)}$ with increasing accuracies.
- Calculation: semidefinite program.
- See:

  G. Tóth, T. Moroder, and O. Gühne, PRL 2015.

Summary

- We considered using semidefinite programs to solve problems in quantum information science.
- We concentrated on problems connected to entanglement theory.

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THANK YOU FOR YOUR ATTENTION!

QUANTERA
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