Entanglement between two spatially separated atomic modes

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Sopron, 23 August 2019

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Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Norwegian Sea
Barents Sea
Reykjavik
ICELAND
Arctic Circle
NORWAY
SWEDEN
FINLAND
RUSSIA
Hammerfest
Tromsø
Murmansk
Kiruna
White Sea
Arkhangelsk
Luleå
Oulu
Umeå
Lake Onega
Trondheim
Gulf of Bothnia
Lake Ladoga
Tórshavn
Faroe Islands (DENMARK)
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Helsinki
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Gulf of Finland
Gävle
ALAND ISLANDS
Bergen
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ESTONIA
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Stavanger
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North Atlantic Ocean
Göteborg
Gotland
Riga
LATVIA
Glasgow
Edinburgh
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DENMARK
Copenhagen
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Baltic Sea
Öland
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Aegean Sea
BALEARIC ISLANDS
Athens
Sevilla
Cagliari
Ionian Sea
Mediterranean Sea
Palermo
Gibraltar (U.K.)
Málaga
Sicily
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Algiers
Tunis
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Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
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Melilla (SPAIN)
Oran
Rabat
Casablanca
TUNISIA
Valletta
MALTA
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300 Miles
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ALGERIA

Outline

Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Full tomography is not possible, we still have to say something meaningful.
- Only collective quantities can be measured.
- Thus, entanglement detection seems to be a good idea ...

Outline

Entanglement

A bipartite state is separable if it can be written as

$$\sum_k p_k \varrho_1^{(a)} \otimes \varrho_2^{(b)}.$$

If a state is not separable then it is entangled.

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

  where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We measure the expectation values $\langle J_l \rangle$.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.
- Symmetric Dicke states with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$. Due to symmetry, $\left\langle \vec{J}^2 \right\rangle$ is maximal. Explicit form:

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right),$$

where $\mathcal{P}_k$ denotes permutations.

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

[photons: Kiesel *et al.,* PRL 2007; Wieczorek *et al.*, PRL 2009; Prevedel *et al.*, PRL 2009.]

[cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Nat. Phys. 2012.]

Dicke states are very interesting because they ...

- ... possess strong multipartite entanglement, like GHZ states.

[GT, JOSAB 2007.]

- ... are optimal for quantum metrology, similarly to GHZ states.

[Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011.]

[GT, PRA 2012;
GT and Apellaniz, J. Phys. A, special issue for “50 year of Bell’s theorem”, 2014.]

- ... are macroscopically entangled, like GHZ states.

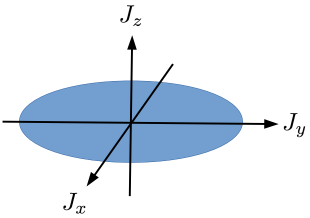
[Fröwis, Dür, PRL 2011]

Collective uncertainties of Dicke states

- Dicke states

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \text{max.},$$
$$\langle J_z^2 \rangle = 0.$$

- "Pancake" like uncertainty ellipse.



Outline

1. **Motivation**
   - Why entanglement is important?
2. **How to detect entanglement in a large ensemble?**
   - Entanglement
   - Collective measurements
3. **Dicke states**
   - Dicke state realized with two-state atoms
4. **Detecting bipartite entanglement of Dicke states**
   - Dicke state in a BEC
   - Entanglement criterion
   - Experimental results

Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.

- Initially all atoms in state $|0\rangle$.

- Dynamics

$$H = a_0^2 a_{+1}^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_{+1} a_{-1}.$$

  Tunneling from mode 0 to the mode $+1$ and -1.

- Understanding the tunneling process

$$|00\rangle \quad \rightarrow \quad \frac{1}{\sqrt{2}}(|1,-1\rangle + |-1,1\rangle) = \text{Dicke state of 2 particles.}$$

Experiment in the group of Carsten Klempt at the University of Hannover II

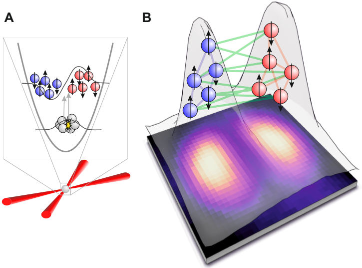
- After some time, we have a state

$$|n_0, n_{-1}, n_{+1}\rangle = |N - 2n, n, n\rangle.$$

- That is, $N - 2n$ particles remained in the 0 state, while $2n$ particles form a symmetric Dicke state.

Experiment in the group of Carsten Klempt at the University of Hannover III

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.



[K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).]

Outline

Very simple entanglement criterion for singlets

- For separable states of two large spins

$$[\Delta(J_x^{(a)}+J_x^{(b)})]^2+[\Delta(J_y^{(a)}+J_y^{(b)})]^2+[\Delta(J_z^{(a)}+J_z^{(b)})]^2 \geq \frac{N}{2}$$

  hold. For singlets, the LHS is zero.

- *Proof.* For product states $|\Psi_a\rangle \otimes |\Psi_b\rangle$

$$\sum_{m=x,y,z}[\Delta(J_m^{(a)}+J_m^{(b)})]^2=\sum_{m=x,y,z}(\Delta J_m^{(a)})^2+\sum_{m=x,y,z}(\Delta J_m^{(a)})^2 \geq \frac{N_a}{2}+\frac{N_b}{2}.$$

  holds.

- True also for separable states due to the concavity of the variance. [GT, Phys. Rev. A (2004).]

Very simple entanglement criterion for Dicke states

- For separable states of two large spins

$$[\Delta(J_{\rm x}^{\rm (a)} - J_{\rm x}^{\rm (b)})]^2 + [\Delta(J_{\rm y}^{\rm (a)} - J_{\rm y}^{\rm (b)})]^2 + [\Delta(J_{\rm z}^{\rm (a)} + J_{\rm z}^{\rm (b)})]^2 \geq \frac{N}{2}.$$

- For Dicke states, the LHS is around $\frac{N}{4}$ for large N, since

$$\begin{aligned}
[\Delta(J_{\rm z}^{\rm (a)} + J_{\rm z}^{\rm (b)})]^2 &= 0, \\
[\Delta(J_{\rm m}^{\rm (a)} + J_{\rm m}^{\rm (b)})]^2 &= {\rm large}, \\
[\Delta(J_{\rm m}^{\rm (a)} - J_{\rm m}^{\rm (b)})]^2 &\approx \frac{N}{8} = {\rm small}
\end{aligned}$$

  for $m = {\rm x, y}$.

- Not a practical criterion since small noise makes the state undetectable, and it assumes symmetry.

Our condition: we use normalized variables

- Normalized variables

$$\tilde{J}_m^{(n)} = \frac{J_m^{(n)}/j_n}{\mathcal{J}^{(n)}},$$

where $m = x, y$ and $n = \mathrm{a}, \mathrm{b}$ (i.e., left well, right well).

- The total spin is

$$j_n = \frac{N_n}{2},$$

and

$$\mathcal{J}^{(n)} = \left\langle \frac{(J_{\mathrm{x}}^{(n)})^2 + (J_{\mathrm{y}}^{(n)})^2}{j_n^2} \right\rangle^{\frac{1}{2}}.$$

- $\mathcal{J}^{(n)} \approx 1$: close to be symmetric. In general, $\mathcal{J}^{(n)} \leq 1$.

The two-well entanglement criterion

Main result

For separable states,

$$\left[(\Delta J_{\mathrm{z}}^{+})^2 + \frac{1}{2}\right] \times \left[\langle(\tilde{J}_{\mathrm{x}}^{-})^2 + (\tilde{J}_{\mathrm{y}}^{-})^2\rangle\right] \geq f\left(\mathcal{J}^{(\mathrm{a})}, \mathcal{J}^{(\mathrm{b})}\right)$$

holds, where $f(x,y) = \frac{(x^2+y^2-1)^2}{xy}$.

Any state violating the inequality is entangled.

Here we define

$$\begin{aligned} J_{\mathrm{z}}^{+} &= J_{\mathrm{z}}^{(a)} + J_{\mathrm{z}}^{(b)}, \\ \tilde{J}_{m}^{-} &= \tilde{J}_{m}^{(a)} - \tilde{J}_{m}^{(b)}. \end{aligned}$$

Correlations for Dicke states

- For the Dicke state

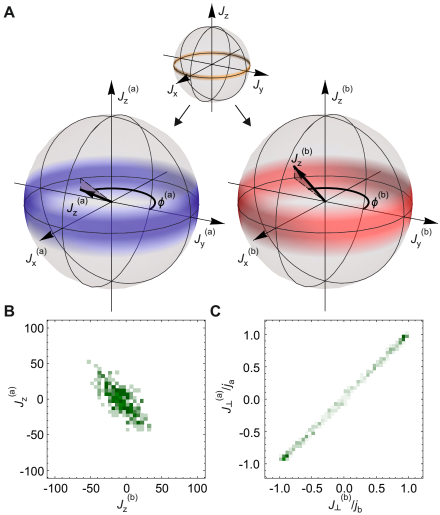
$$
\begin{aligned}
(\Delta(J_x^{(a)} - J_x^{(b)}))^2 &\approx 0, \\
(\Delta(J_y^{(a)} - J_y^{(b)}))^2 &\approx 0, \\
(\Delta J_z)^2 &= 0.
\end{aligned}
$$

- Measurement results on well "b" can be predicted from measurements on "a"

$$
\begin{aligned}
J_x^{(b)} &\approx J_x^{(a)}, && \leftarrow \text{correlated} \\
J_y^{(b)} &\approx J_y^{(a)}, && \leftarrow \text{anticorrelated} \\
J_z^{(b)} &= -J_z^{(a)}. && \leftarrow \text{anticorrelated}
\end{aligned}
$$

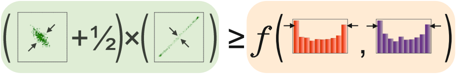
Outline

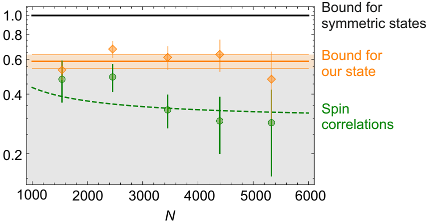
Correlations for Dicke states - experimental results



Here, $J_\perp^{(n)} = \cos\alpha J_x^{(n)} + \sin\alpha J_y^{(n)}$.

Violation of the criterion: entanglement is detected



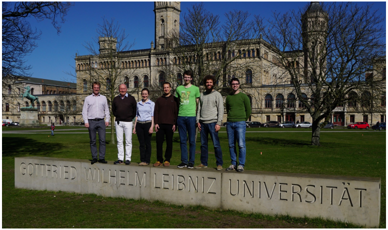


For separable states,

$$\left[(\Delta J_z^+)^2 + \frac{1}{2}\right] \times \left[\langle (\tilde{J}_x^-)^2 + (\tilde{J}_y^-)^2 \rangle\right] \geq f\left(\mathcal{J}^{(a)}, \mathcal{J}^{(b)}\right)$$

holds, where $f(x, y) = \frac{(x^2+y^2-1)^2}{xy}$.

Collaborators on entanglement conditions for Dicke states



C. Klempt, I. Kruse, J. Peise,
K. Lange, B. Lücke

Hannover

G. Vitagliano

I. Apellaniz

M. Kleinmann

Bilbao (G.T.)

Summary

- Detection of bipartite entanglement close to Dicke states.

K. Lange, J. Peise, B. Lücke, I. Kruse,
G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt,
Entanglement between two spatially separated atomic modes,
Science 360, 416 (2018).

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Appendix

Proof

Product states. For states of the form $|\Psi^{(\mathrm{a})}\rangle \otimes |\Psi^{(\mathrm{b})}\rangle$.

$$\left[(\Delta J_{\mathrm{z}}^{+})^2 + \frac{1}{2}\right] \times \left[(\Delta \tilde{J}_{\mathrm{x}}^{-})^2 + (\Delta \tilde{J}_{\mathrm{y}}^{-})^2\right]$$
$$= [(\mathcal{U}^{(\mathrm{a})} + \tfrac{1}{4}) + (\mathcal{U}^{(\mathrm{b})} + \tfrac{1}{4})] \cdot (\mathcal{V}^{(\mathrm{a})} + \mathcal{V}^{(\mathrm{b})})$$
$$\geq 4\sqrt{(\mathcal{U}^{(\mathrm{a})} + \tfrac{1}{4})(\mathcal{U}^{(\mathrm{b})} + \tfrac{1}{4})\mathcal{V}^{(\mathrm{a})}\mathcal{V}^{(\mathrm{b})}} \geq 1$$

holds, where we used the notation

$$\mathcal{U}^{(n)} = (\Delta J_{\mathrm{z}}^{(n)})^2, \qquad \mathcal{V}^{(n)} = (\Delta \tilde{J}_{\mathrm{x}}^{(n)})^2 + (\Delta \tilde{J}_{\mathrm{y}}^{(n)})^2$$

for $n = \mathrm{a}, \mathrm{b}$. We used that
(i) $[\Delta(A^{(\mathrm{a})} + A^{(\mathrm{b})})]^2 = (\Delta A^{(\mathrm{a})})^2 + (\Delta A^{(\mathrm{b})})^2$,
(ii) Inequality between the arithmetic and the geometric mean,
(iii) Our number-phase like uncertainty.

Proof II

Using $\langle(\tilde{J}_x^{(n)})^2\rangle + \langle(\tilde{J}_y^{(n)})^2\rangle = 1$ for $n = \mathrm{a}, \mathrm{b}$, our inequality for product states yields

$$2\left[(\Delta J_z^+)^2 + \frac{1}{2}\right](\mathcal{S} - \mathcal{C}) \geq \mathcal{S},$$

where correlations between the two subsystems are characterized by

$$\mathcal{C} = \left\langle \frac{J_x^{(a)} J_x^{(b)} + J_y^{(a)} J_y^{(b)}}{j_a j_b} \right\rangle,$$

and

$$\mathcal{S} = \mathcal{J}^{(a)} \mathcal{J}^{(b)}.$$

$\mathcal{C}$ can be negative and $|\mathcal{C}| \leq \mathcal{S}$.

The normalization with the total spin will make it easier to adapt our criterion to experiments with a varying particle number in the ensembles.

Proof III

Separable states. We now consider a mixed separable state of the form $\varrho_{\rm sep} = \sum_k p_k |\Psi_k^{\rm (a)}\rangle \otimes |\Psi_k^{\rm (b)}\rangle$. For such states, we can write the following series of inequalities

$$2\left[(\Delta J_{\rm z}^{+})^2 + \frac{1}{2}\right](\mathcal{S} - \mathcal{C}) \geq 2\left[\sum_k p_k (\Delta J_{\rm z})^2{}_k + \frac{1}{2}\right]\left[\sum_k p_k \left(\mathcal{S}_k - \mathcal{C}_k\right)\right]$$
$$\geq 2\left[\sum_k p_k \sqrt{\left((\Delta J_{\rm z})^2{}_k + \frac{1}{2}\right)(\mathcal{S}_k - \mathcal{C}_k)}\right]^2 \geq \left(\sum_k p_k \sqrt{\mathcal{S}_k}\right)^2,$$

Subscript k refers to the $k^{\rm th}$ sub-ensemble $|\Psi_k^{\rm (a)}\rangle \otimes |\Psi_k^{\rm (b)}\rangle$.
(i) The first inequality in is due to $(\Delta J_{\rm z}^{+})^2$ and $\mathcal{S}$ being concave in the quantum state.
(ii) The second inequality is based on the Cauchy-Schwarz inequality.
(iii) The third inequality is the application of the previous inequality for all sub-ensembles.

Proof IV

Next, we find a lower bound on the RHS of the last inequality based on the knowledge of $\mathcal{J}^{(\mathrm{a})}$ and $\mathcal{J}^{(\mathrm{b})}$. We find that

$$\sum_k p_k \left(\mathcal{J}_k^{(\mathrm{a})}\mathcal{J}_k^{(\mathrm{b})}\right)^{1/2} \geq (\mathcal{J}^{(\mathrm{a})})^2 + (\mathcal{J}^{(\mathrm{b})})^2 - 1,$$

which is based on noting $(xy)^{1/4} \geq x + y - 1$ for $0 \leq x, y \leq 1$.

Using this to bound the RHS from below and dividing by $\mathcal{S}$ we obtain

$$\left[(\Delta J_{\mathrm{z}}^{+})^2 + \frac{1}{2}\right] \times \left[2 - 2\frac{\mathcal{C}}{\mathcal{S}}\right] \geq \frac{\left[(\mathcal{J}^{(\mathrm{a})})^2 + (\mathcal{J}^{(\mathrm{b})})^2 - 1\right]^2}{\mathcal{S}}.$$