Quantum entanglement and its use in metrology

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Budapest, 2 November 2021

Outline

Motivation

- There have been many experiments recently aiming to create multiparticle quantum states.
- Quantum Information Science can help to find good targets for such experiments.
- Highly entangled multiparticle quantum states are good candidates for such experiments.
- Such states are needed in quantum metrology.

Outline

Theory of quantum entanglement

- Statistical physics: the ground state of spin systems can be factorized or not factorized.
- There are no pure states in an experiment. The product states must be generalized to the case of mixed states.
- Separable states = a mixture of product states.
- Entangled (non-separable) states are useful in certain quantum information processing applications.

Separable states

A state of N-particles is fully separable if it can be written in the following form

$$\varrho_{\rm sep} = \sum_m p_m \rho_m^{(1)} \otimes \rho_m^{(2)} \otimes ... \otimes \rho_m^{(N)},$$

where $\rho_m^{(n)}$ are single-particle pure states.

- Separable states are essentially states that can be created without interaction between the particles by simply mixing the product states.

Separable states II

- Let us have two separable states

$$\varrho_{\mathrm{sep},k} = \sum_m p_{m,k}\rho_{m,k}^{(1)} \otimes \rho_{m,k}^{(2)} \otimes ... \otimes \rho_{m,k}^{(N)}$$

for $k = 1, 2$.

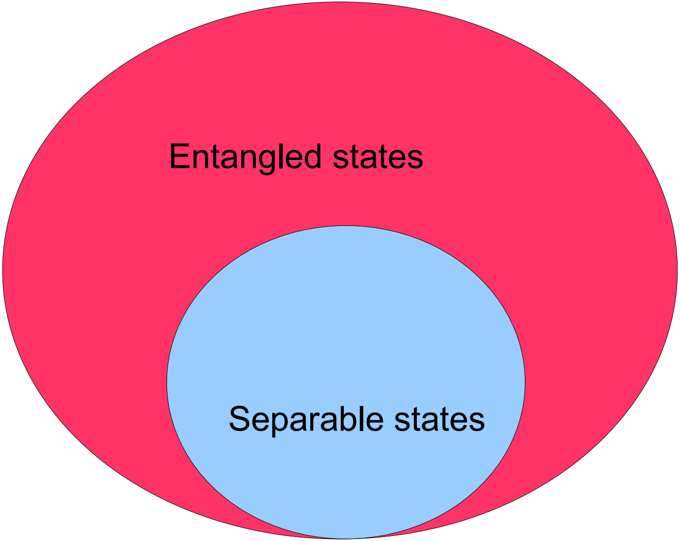
- Their mixture

$$\varrho = p\varrho_{\mathrm{sep},1} + (1-p)\varrho_{\mathrm{sep},2},$$

where $0 \leq p \leq 1$, is also a separable state.

- Thus, the set of separable states is convex.

Convex sets



Many-body entanglement

A pure state *k-producible*, if it can be written as

$$|\Psi\rangle = \otimes_m |\psi_m\rangle,$$

where $|\psi_m\rangle$ are many-particle states with at most $k_m \leq k$ particles.

A mixed state is k-producucble if it can be written as a mixture of k-producible states.

A state that is not k-prodcuble, is at least $(k+1)-$particle entangled.

O. Gühne and G. Tóth, NJP 2005.

Genuine multipartite entanglement

- Genuine multipartite entanglement=N-particle entanglement in a system of N particles.
- Biseparable states=states that are not genuine multipartite entangled.

Examples

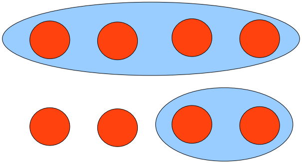
Examples

Two entangled states of four qubits:

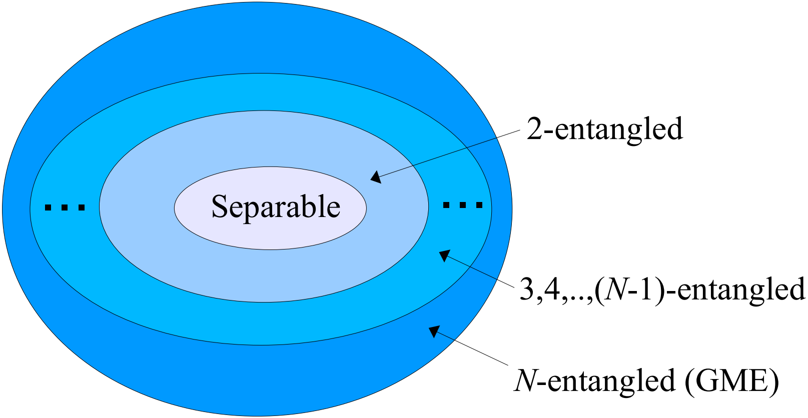
$$|\mathrm{GHZ}_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$

$$|\Psi_B\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |0011\rangle) = \frac{1}{\sqrt{2}}|00\rangle \otimes (|00\rangle + |11\rangle).$$

- The first state is genuine multipartite entangled, and 4-entangled.
- The second state is biseparable, and 2-entangled.



Convex sets



Convex sets of quantum states

Outline

Entanglement detection

Looking at the definition

$$\varrho_{\rm sep} = \sum_m p_m \rho_m^{(1)} \otimes \rho_m^{(2)} \otimes ... \otimes \rho_m^{(N)},$$

we see that is a difficult task to decide whether a quantum state is entangled or not.

There are no general methods.

Entanglement witness

An operatior W is an entanglement witness, if the following conditions are fulfilled

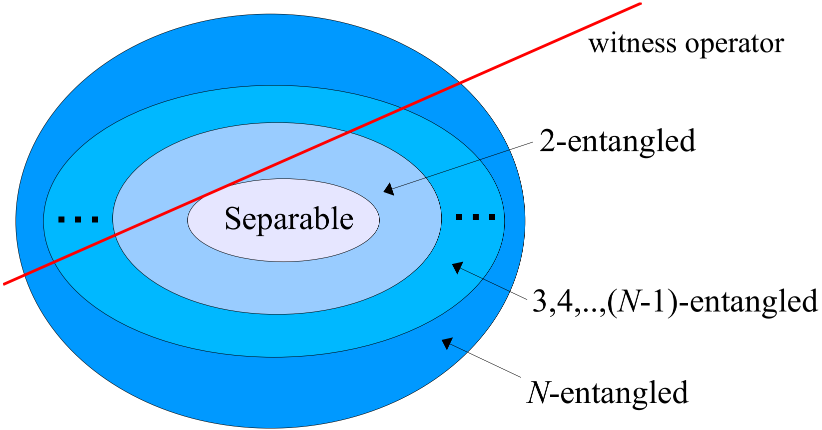
1. For separable states it is non-negative

$$\langle W \rangle_{\varrho_{\rm sep}} \equiv {\rm Tr}(\varrho_{\rm sep} W) \geq 0.$$

2. There is such an entangled state $\varrho_{\rm ent}$ for which the expectation value is negative

$$\langle W \rangle_{\varrho_{\rm ent}} < 0.$$

Entanglement witness II



Entanglement witness III

- We have to look for witness operators that are easy to measure.
- Or, if we consider a general witness operator, it might be a highly nonlocal operator.
- We cannot measure its expectation value directly.
- It must be decomposed it as

$$W = \sum_k A_k^{(1)} \otimes A_k^{(2)} \otimes ... \otimes A_k^{(N)}.$$

- The expectation value can be obtained as

$$\langle W \rangle \equiv \mathrm{Tr}(\varrho W) = \sum_k \left\langle A_k^{(1)} \otimes A_k^{(2)} \otimes ... \otimes A_k^{(N)} \right\rangle.$$

Outline

Detecting entanglement with the Hamiltonian of spin chains

For a separable state of two qubits

$$-1 \leq \langle\sigma_x^{(1)}\sigma_x^{(2)}\rangle + \langle\sigma_y^{(1)}\sigma_y^{(2)}\rangle + \left\langle\sigma_z^{(1)}\sigma_z^{(2)}\right\rangle \leq 1.$$

holds.

- Basic idea: we take the maximum for product states of the type $|\psi_1\rangle \otimes |\psi_2\rangle$.
- For such states
$$\langle\sigma_l^{(1)}\sigma_l^{(2)}\rangle = \langle\sigma_l^{(1)}\rangle\langle\sigma_l^{(2)}\rangle$$
for $l = x, y, z$.

Detecting entanglement with the Hamiltonian of spin chains II

- The expectation value can be written as a scalar product

$$\langle\sigma_x^{(1)}\sigma_x^{(2)}\rangle + \langle\sigma_y^{(1)}\sigma_y^{(2)}\rangle + \left\langle\sigma_z^{(1)}\sigma_z^{(2)}\right\rangle = \vec{s}_1 \cdot \vec{s}_2,$$

  where

$$\vec{s}_n = \left(\begin{array}{c} \langle\sigma_x^{(n)}\rangle_{|\psi_n\rangle} \\ \langle\sigma_y^{(n)}\rangle_{|\psi_n\rangle} \\ \langle\sigma_z^{(n)}\rangle_{|\psi_n\rangle} \end{array} \right)$$

  for $n = 1, 2$.
- The Cauchy-Schwarz yields

$$|\vec{s}_1 \vec{s}_2| \leq |\vec{s}_1||\vec{s}_2| = 1.$$

- The bound is also valid for separable states, since they are the mixture of product states.

Detecting entanglement with the Hamiltonian of spin chains III

- For the singlet state

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

we have

$$\langle\sigma_x^{(1)}\sigma_x^{(2)}\rangle + \langle\sigma_y^{(1)}\sigma_y^{(2)}\rangle + \left\langle\sigma_z^{(1)}\sigma_z^{(2)}\right\rangle = -3.$$

- An entanglement witness can be written as

$$W = \mathbb{1} + \sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)} + \sigma_z^{(1)}\sigma_z^{(2)}$$

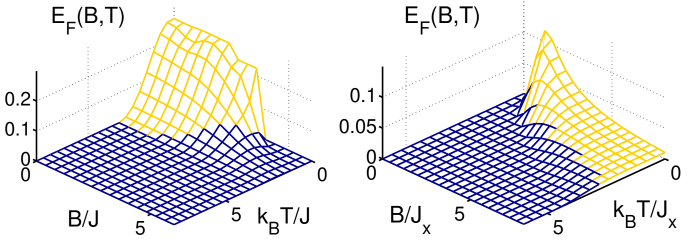
for which

$$\langle W\rangle = -2.$$

Spin chains

- If $\langle H \rangle$ is smaller than the energy minium of the classical latticem then the system is entangled.
- Antiferromagnetic Heisenberg-Hamiltonian operator with periodic boundary condition on d-dimensional square lattice.
- XY-Hamiltonian operator with periodic boundary condition on d-dimensional square lattice.

Numerical results



(a) (b)

(a) Heisenberg chain, 8 spins.

(b) Ising chain

E_F=entanglement of formation for the nearest neighbors

Yellow=detected states.
We detect the states that have more than minimal entanglement.

GT, Phys. Rev. A 2005.

Outline

Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.

- Symmetric Dicke states with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N^{(N/2)}\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

Due to symmetry, $\left\langle \vec{J}^2 \right\rangle$ is maximal.

- E.g., for four qubits they look like

$$|D_4^{(2)}\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, and H. Weinfurter, PRL 2007; Prevedel *et al.*, PRL 2007;
W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Science 2011

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like Greenberger-Horne-Zelinger states (GHZ, $\approx$ Schrödinger cat states)

  GT, JOSAB 2007.

- ... are optimal for quantum metrology, similarly to GHZ states.

  Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011.
  GT, PRA 2012;
  GT and Apellaniz, J. Phys. A, special issue for "50 year of Bell's theorem", 2014.

- ... are macroscopically entangled, like GHZ states.

  Fröwis, Dür, PRL 2011.

Entanglement detection close to Dicke states

For biseparable (not genuine multipartite entangled) ρ

$$F_{D_N} = \mathrm{Tr}(\rho|D_N\rangle\langle D_N|) \leq \frac{1}{2}\frac{N}{N-1} =: C.$$

Any state that violates the above inequality has true multibody entanglement. (We omit the $N/2$ superscript.)

- For large N, $C \approx 1/2$. That is, only a fidelity of $1/2$ is required for a successful experiment.
- The limit cannot be smaller than $1/2$.
- Previously, it was only known for GHZ and cluster states that this limit was $1/2$.

GT, JOSAB 2007.

Decomposition of the projector I

- The fidelity with respect to the Dicke state is

$$F_{D_N} = \mathrm{Tr}(\rho |D_N\rangle\langle D_N|).$$

- The witness is

$$W = \frac{1}{2}\frac{N}{N-1}\mathbb{1} - |D_N\rangle\langle D_N|.$$

- We have to decompose the projector

$$|D_N\rangle\langle D_N| = \sum_k A_k \otimes A_k \otimes ... \otimes A_k.$$

- We did this for $N = 6$, for which

$$W = 0.6\mathbb{1} - |D_N\rangle\langle D_N|.$$

Decomposition of the projector II

$$
\begin{aligned}
64|D_6^{(3)}\rangle\langle D_6^{(3)}| = & -0.6[\mathbb{1}]+0.3[x\pm\mathbb{1}]-0.6[x]+0.3[y\pm\mathbb{1}]-0.6[y]+0.2[z\pm\mathbb{1}]-0.2[z] \\
& +0.2\mathrm{Mermin}_{0,z}+0.05[x\pm y\pm\mathbb{1}]-0.05[x\pm z\pm\mathbb{1}]-0.05[y\pm z\pm\mathbb{1}] \\
& -0.05[x\pm y\pm z]+0.2[x\pm z]+0.2[y\pm z]+0.1[x\pm y] \\
& +0.6\mathrm{Mermin}_{x,z}+0.6\mathrm{Mermin}_{y,z}. \qquad (31)
\end{aligned}
$$

Here we use the notation $[x+y]=(\sigma_x+\sigma_y)^{\otimes 6}$, $[x+y+\mathbb{1}]=(\sigma_x+\sigma_y+\mathbb{1})^{\otimes 6}$, etc. The $\pm$ sign denotes a summation over the two signs, i.e., $[x\pm y]=[x+y]+[x-y]$. The Mermin operators are defined as

$$
\mathrm{Mermin}_{a,b} := \sum_{k\ \mathrm{even}} (-1)^{k/2} \sum_k \mathcal{P}_k(\otimes_{i=1}^k \sigma_a \otimes_{i=k+1}^N \sigma_b), \qquad (32)
$$

where $\sigma_0=\mathbb{1}$. That is, it is the sum of terms with even number of σ_a's and σ_b's, with the sign of the terms depending on the number of σ_a's. The expectation value of the operators $\mathrm{Mermin}_{a,b}$ can be measured based on the decomposition [23]

$$
\mathrm{Mermin}_{a,b} = \frac{2^{N-1}}{N}\sum_{k=1}^{N}(-1)^k\left[\cos\left(\frac{k\pi}{N}\right)a+\sin\left(\frac{k\pi}{N}\right)b\right]^{\otimes N}. \qquad (33)
$$

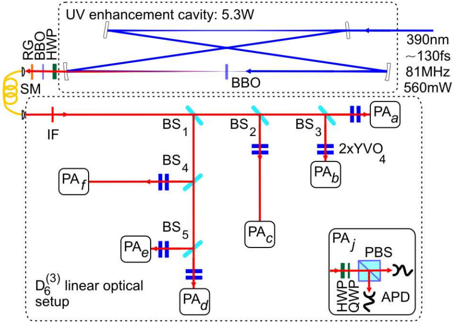
GT, W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, and H. Weinfurter, New J. Phys. 2009.

Results

settings [15,16]. We have determined $F_{D_6^{(3)}} = 0.654 \pm 0.024$ with a measurement time of 31.5 h. This allows the application of the generic entanglement witness [10] $\langle \mathcal{W}_g \rangle = 0.6 - F_{D_6^{(3)}} = -0.054 \pm 0.024$ and thus proves genuine six-qubit entanglement of the observed state with a significance of 2 standard deviations (Fig. 4).

C. Schwemmer, GT, A. Niggebaum, T. Moroder, D. Gross, O. Gühne, and H. Weinfurter, Efficient Tomographic Analysis of a Six Photon State, PRL 2014.

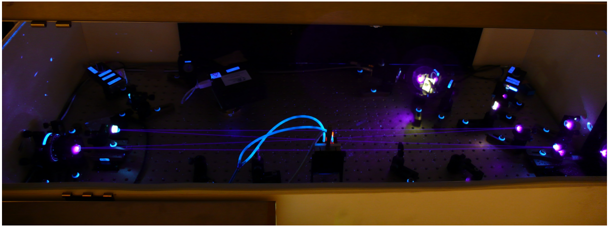
Experiments with photons

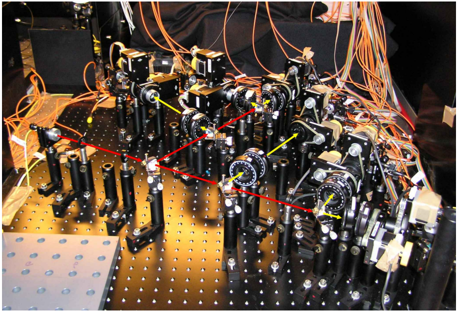


MPQ, München, experiment with six photons

$$|D_6^{(3)}\rangle = \frac{1}{\sqrt{20}}\left(|111000\rangle + |110100\rangle + \ldots + |000111\rangle\right).$$

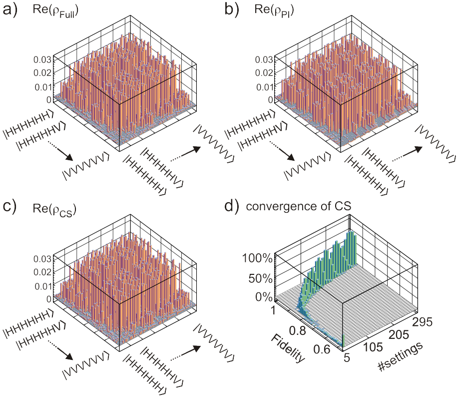
Experiments with photons





Experiments with photons

State tomography of the six-photon state



C. Schwemmer, G. Tóth, A. Niggebaum, T. Moroder, D. Gross, O. Gühne, and H. Weinfurter, Efficient Tomographic Analysis of a Six Photon State, PRL 2014.

Outline

Many-particle systems

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the

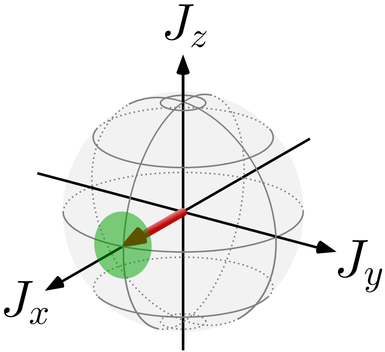
$$(\Delta J_l)^2 := \langle J_l^2\rangle - \langle J_l\rangle^2$$

variances.

Fully polarized state

- State fully polarized in the x-direction

$$|+1/2\rangle_x^{\otimes N}.$$



We thank I. Appelaniz for the figure.

Spin squeezing

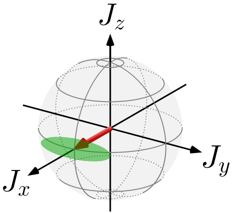
Definition

Uncertainty relation for the spin coordinates

$$(\Delta J_y)^2(\Delta J_z)^2 \geq \tfrac{1}{4}|\langle J_x\rangle|^2.$$

If $(\Delta J_z)^2$ is smaller than the standard quantum limit $\frac{1}{2}|\langle J_x\rangle|$ then the state is called spin squeezed (mean spin in the x direction!).

M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).



We thank I. Appelaniz for the figure.

Spin squeezing II

Definition

Spin squeezing criterion for the detection of quantum entanglement

$$\frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2} \geq \frac{1}{N}.$$

If a quantum state violates this criterion then it is entangled.

- Application: Quantum metrology, magnetometry. Used many times in experiments.

  A. Sørensen *et al.*, Nature **409**, 63 (2001); experiments by E. Polzik, M.W. Micthell with cold atomic ensembles; M. Oberthaler, Ph. Treutlein with Bose-Einsetein condensates.

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_{\rm x} \rangle, \langle J_{\rm y} \rangle, \langle J_{\rm z} \rangle),$$
$$\vec{K} := (\langle J_{\rm x}^2 \rangle, \langle J_{\rm y}^2 \rangle, \langle J_{\rm z}^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

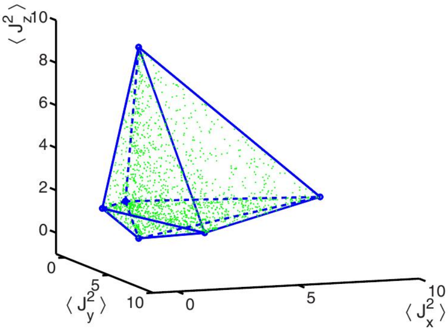
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2},$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2},$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
spin-j: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality. For separable states

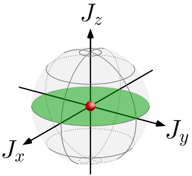
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_z)^2$$

holds.

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2} + 1\right) = \max.,$$
$$\langle J_z^2 \rangle = 0.$$

- "Pancake" like uncertainty ellipse.



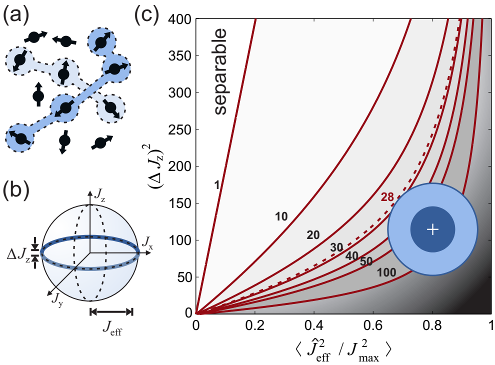
Bose-Einstein condensate people

Netflix movie "Spectral"

Filmed in Budapest

Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.

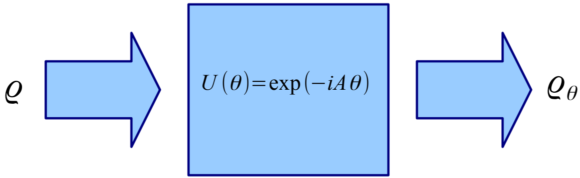


$$J_{\text{eff}}^2 = J_x^2 + J_y^2, \quad J_{\max} = \frac{N}{2}.$$

B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, GT, and C. Klempt, PRL 112, 155304 (2014).

Outline

Multipartite entanglement and quantum metrology



- Cramér-Rao bound

$$(\Delta\theta)^2 \geq \frac{1}{\nu F_Q[\varrho, A]},$$

where ν is the number of independent repetitions.

- Quantum Fisher information

$$F_Q[\varrho, A] = 2\sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |A_{ij}|^2.$$

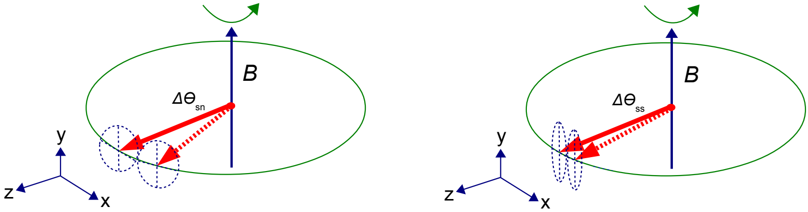
- Here λ_i denotes the eigenvalues of the density matrix, A_{ij} are the matrix elelements of A in the eigenbasis of the density matrix.

Special case $A = J_l$: linear interferometer

- The operator A is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$

- Magnetometry with a linear interferometer



The quantum Fisher information vs. entanglement

- Shot-noise limit: For separable states

$$F_Q[\varrho, J_l] \leq N, \qquad (\Delta\theta)^2 \geq \frac{1}{\nu N}, \qquad l = x, y, z.$$

Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010).

- Heisenberg limit: For entangled states

$$F_Q[\varrho, J_l] \leq N^2, \qquad (\Delta\theta)^2 \geq \frac{1}{\nu N^2}, \qquad l = x, y, z.$$

where the bound can be saturated.

Multipartite entanglement and Quantum Fisher information

For N-qubit k-producible states states, the quantum Fisher information is bounded from above by

$$F_Q[\varrho, J_l] \leq sk^2 + (N - sk)^2,$$

where

$$s = \lfloor \tfrac{N}{k} \rfloor,$$

and $\lfloor \frac{N}{k} \rfloor$ denotes the integer part of $\frac{N}{k}$.

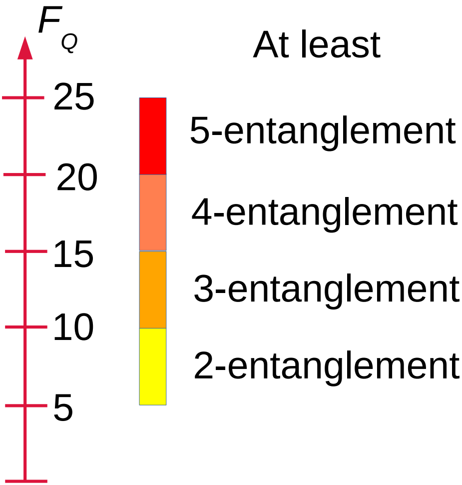
Simpler form with a bound that is not optimal

$$F_Q[\varrho, J_l] \leq Nk, \qquad (\Delta\theta)^2 \geq \frac{1}{\nu Nk}.$$

Hyllus *et al.*, PRA 2012; GT PRA 2012 (back-to-back).

Multipartite entanglement and Quantum Fisher information II

5 spin-1/2 particles



Metrology with Dicke states

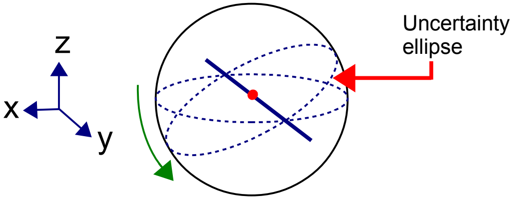
- For our symmetric Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Measure $\langle J_z^2 \rangle$ to estimate θ. (We cannot measure first moments, since they are zero.)



Metrology with Dicke states

- They found that

$$F_Q[\varrho, J_l] \leq N,$$

is violated since they measured that

$$(\Delta\theta)^2 < \frac{1}{\nu N}.$$

Metrology with cold gases: B. Lücke, M Scherer, J. Kruse, L. Pezze, F. Deuretzbacher, P. Hyllus, O. Topic, J. Peise, W. Ertmer, J. Arlt, L. Santos, A. Smerzi, C. Klempt, Science 2011.

Metrology with photons: R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, P. Hyllus, L. Pezze, A. Smerzi, PRL 2011.

Summary

- Effective entanglement detection is very important in an experiment.
- We have presented methods that can be used in systems where the particles are addressable separately.
- We also present methods that detect entanglement in multiparticle systems.
- We also found entanglement criteria based on quantum metrology.

www.gtoth.eu

THANK YOU FOR YOUR ATTENTION!

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