Witnessing metrologically useful multiparticle entanglement

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Why multipartite entanglement and metrology are important?

- Full tomography is not possible, we still have to say something meaningful.
- Claiming “entanglement” is not sufficient for many particles.
- We should tell
  - How entangled the state is
  - What the state is good for, etc.

Outline

1. Introduction and motivation
2. **Quantum metrology**
   - Setting the scene
   - The quantum Fisher information
3. Witnessing metrological usefulness
   - Obtaining bounds on the quantum Fisher information
   - Applications of our method

Basic notions

- Collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$, $\sigma_l^{(k)}$ are Pauli spin matrices, and N is the number of spin-$\frac{1}{2}$ particles.

- A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes ... \otimes \varrho_k^{(N)}.$$

If a state is not separable then it is entangled (Werner, 1989).

- Similar definitions for multipartite entanglement.

k-producibility/k-entanglement

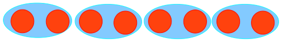
A pure state is k-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

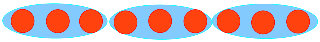
where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., O. Gühne and GT, New J. Phys 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.



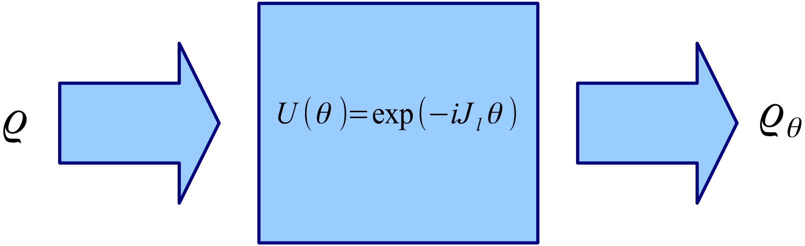
two-producible



three-producible

Quantum metrology

- Fundamental task in metrology with a linear interferometer



- We have to estimate θ in the dynamics

$$U = \exp(-iJ_l\theta)$$

where $l \in \{x, y, z\}$.

Outline

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}.$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is given by an explicit formula for ϱ and A as

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

- Linear interferometer: $A = J_l$, $l = x, y, z$.

The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \leq N.$$

[Pezze, Smerzi, PRL 2009]

- For states with at most k-particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \leq kN.$$

[Hyllus *et al.*, PRA 2012; GT, PRA 2012].

- If a state violates the above inequality then it has $(k+1)$-particle metrologically useful entanglement.

Metrological precision vs. entanglement

- For separable states

$$(\Delta\theta)^2 \geq \frac{1}{N}.$$

[Pezze, Smerzi, PRL 2009]

- For states with at most k-particle entanglement (k is divisor of N)

$$(\Delta\theta)^2 \geq \frac{1}{kN}.$$

[Hyllus *et al.*, PRA 2012; GT, PRA 2012].

- If a state violates the above inequality then it has $(k+1)$-particle metrologically useful entanglement.

Metrologically useful entanglement vs. general entanglement

- Not all entangled states are more useful than separable states.
  [Hyllus, Gühne, Smerzi, PRA 2010]

- For states with metrologically useful k-particle entanglement $\varrho_{k-\mathrm{ent}}$

$$F_Q[\varrho_{k-\mathrm{ent}}, J_z] > F_Q[\varrho_{(k-1)-\mathrm{ent}}, J_z]$$

  holds for all $\varrho_{(k-1)-\mathrm{ent}}$.

- States with metrologically useful k-particle entanglement are more useful than the state

$$|\mathrm{GHZ}_{(k-1)}\rangle \otimes |\mathrm{GHZ}_{(k-1)}\rangle \otimes |\mathrm{GHZ}_{(k-1)}\rangle \otimes ....$$

Outline

Witnessing metrological usefulness

- Direct measurement of the sensitivity
  - Measure $(\Delta\theta)^2$.
  - Obtain bound on F_Q and multipartite entanglement.
  - Experimentally challenging, since we need dynamics.
  - The precision is affected by the noise during the dynamics.

  [Experiments in cold atoms by the groups of Oberthaler, Klempt; photonic experiments of the Weinfurter group.]

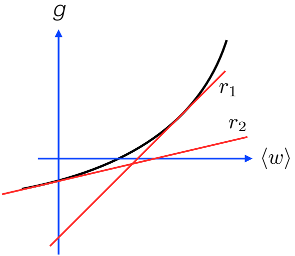
- Witnessing (our choice)
  - Estimate how good the precision were, if we did the metrological process.
  - Assume a perfect metrological process. Characterizes the state only.
  - Another approach for thermal states
    [P. Hauke, M. Heyl, L. Tagliacozzo, and P. Zoller, Nat. Phys. (2016)]

Legendre transform (used often in physics)

- Optimal linear lower bound on a convex function $g(\varrho)$ based on an operator expectation value $w = \langle W \rangle_\varrho = \mathrm{Tr}(W\varrho)$

$$g(\varrho) \geq rw - const.,$$

where $w = \mathrm{Tr}(\varrho W)$.



- For every slope r there is a "*const.*"
- Textbooks say

$$g(\varrho) \geq \mathcal{B}(w) := rw - \hat{g}(rW),$$

where $\hat{g}$ is the Legendre transform.

Legendre transform II

- Bound is best if we optimize over r as

$$g(\varrho) \geq \mathcal{B}(w) := \sup_r \left[rw - \hat{g}\left(rW\right)\right],$$

where again $w = \mathrm{Tr}(\varrho W)$.

- Then, we need the Legendre transform is given as

$$\hat{g}(W) = \sup_\varrho [\langle W \rangle_\varrho - g(\varrho)].$$

- Used for estimating entanglement measures, optimization over ϱ or Ψ is needed.

[Gühne, Reimpell, Werner, PRL 2007; Eisert, Brandao, Audenaert, NJP 2007.]

- Related approach: Lagrange multipliers. Used also in quantum information.

Bounding the quantum Fisher information

Problem: $g(\varrho) = F_Q$ and the Legendre transform is an optimization over all elements of ϱ!

Key observation: F_Q is the convex roof of the variance.
[GT, Petz, PRA 2013; S. Yu, arXiv1302.5311 (2013);
GT, Apellaniz, J. Phys. A: Math. Theor. 2014]

Hence, the Legendre transform given as an optimization over a single (!) real variable

$$\hat{\mathcal{F}}_{\mathrm{Q}}(W) = \sup_{\mu} \left\{ \lambda_{\max} \left[W - 4(J_l - \mu)^2 \right] \right\}.$$

Bounding the quantum Fisher information II

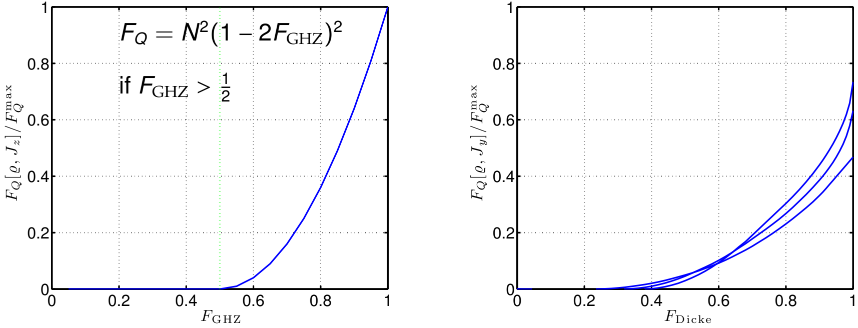
Big surprise

The quantum Fisher information is the ideal quantity for using the Legendre transform technique.

Outline

Witnessing the quantum Fisher information based on the fidelity

- Bound the quantum Fisher information based on some measurements. First, consider small systems.
  [See also Augusiak *et al.*, 1506.08837.]



Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for $N = 4, 6, 12$.

[Apellaniz *et al.*, arXiv:1511.05203.]

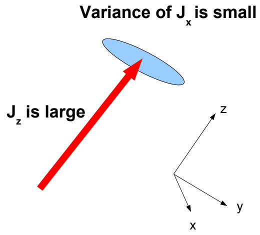
Bounding the qFi based on collective measurements

Bound on the quantum Fisher information for spin squeezed states (Pezze-Smerzi bound)

$$F_Q[\varrho, J_y] \geq \frac{\langle J_z \rangle^2}{(\Delta J_x)^2}.$$

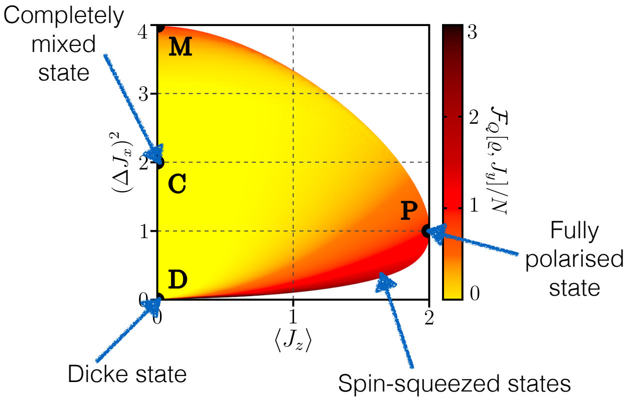
[Pezze, Smerzi, PRL 2009.]

- States with a large F_Q are spin-squeezed states:



Bounding the qFi based on collective measurements II

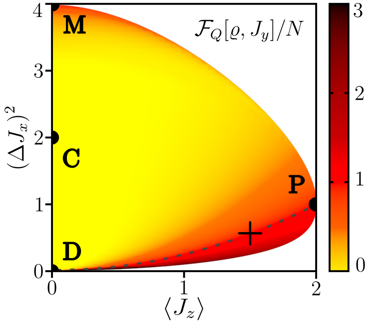
- Optimal bound for the quantum Fisher information $F_Q[\varrho, J_y]$ for spin squeezing for $N = 4$ particles

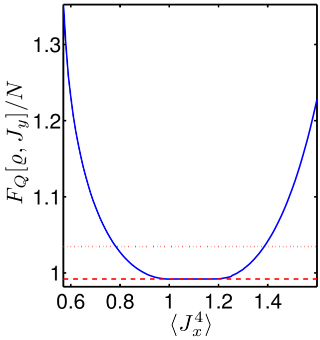


[Apellaniz, Kleinmann, Gühne, GT, arXiv:1511.05203.]

Bounding the qFi based on collective measurements III

- The bound can be obtained if additional expectation value, i.e., $\langle J_x^2 \rangle$ is measured, or we assume symmetry:





[Apellaniz, Kleinmann, Gühne, GT, arXiv:1511.05203.]

Spin-squeezing experiment

- Experiment with $N = 2300$ atoms,

$$\xi_s^2 = N\frac{(\Delta J_x)^2}{\langle J_z\rangle^2} = -8.2\text{dB} = 10^{-8.2/10} = 0.1514.$$

[Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 2010.]

- The Pezze-Smerzi bound is:

$$\frac{\mathcal{F}_Q[\varrho_N, J_y]}{N} \geq \frac{1}{\xi_s^2} = 6.605 \quad \rightarrow \text{7-qubit entanglement}$$

- We get the same value for our method!

[Pezze, Smerzi, PRL 2009]

- Similar calculations for Dicke state experiments!

[Lücke, Peise, Vitagliano, Arlt, Santos, Tóth, Klempt, PRL 2014.]

Our method is unique for entanglement detection in large systems

- Advantages of our approach
  - It works for any set of observables to be measured.
  - It quantifies multipartite entanglement, not only detects it.
  - It works for large systems (i.e., 2000×2000 density matrix).
- Alternatives that come close
  - Work for large systems, but for a specific set of observables, e.g., spin squeezing inequalities.
  - Work for arbitrary observables, but only for small systems, e.g., methods using semidefinite programming.

Summary

- We discussed a **very flexible** method to detect multipartite entanglement and metrological usefulness.
- We can choose a set of operators and the method gives an optimal lower bound on F_Q.

Apellaniz, Kleinmann, Gühne, Tóth, arxiv: arXiv:1511.05203
[See also Apellaniz, Lücke, Peise, Klempt, GT, NJP 17, 083027 (2015)]

THANK YOU FOR YOUR ATTENTION!

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