Criteria for detecting entanglement close to Dicke states with many-body correlations

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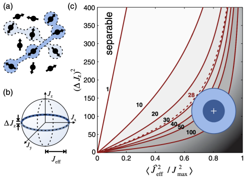
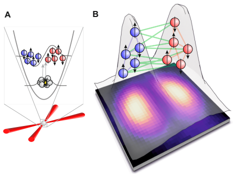
Motivation

- There have been successful experiments in detecting multipartite and bipartite entanglement in Dicke states of many particles.





Lücke PRL 2014, Vitagliano NJP 2017; Lange Science 2018, Vitagliano Quantum 2023

- They need measuring the collective observables J_x, J_y and J_z.
- The resolution of the particle number detection is not 1 particle. It can be for instance ~ 10.
- Particle-number resolving detection could improve the detected quality of the state dramatically.
- We could also have new entanglement criteria relying on single particle resolution.

Outline

Entanglement - Mixed states

Definition

A quantum state is called separable if it can be written as a convex sum of product states as [Werner, 1989]

$$\varrho = \sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)},$$

where p_k form a probability distribution ($p_k > 0, \sum_k p_k = 1$), and $\varrho_n^{(k)}$ are single-qudit density matrices.

A state that is not separable is called entangled.

- We cannot always decide whether the state is entangled.

k-producibility/k-entanglement

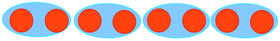
A pure state is k-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

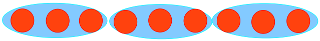
where $|\Phi_l\rangle$ are states of at most k particles.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
e.g., Gühne, GT, NJP 2005.

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.

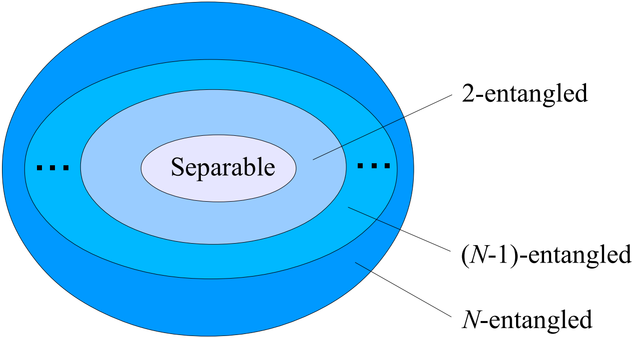


2-entangled



3-entangled

k-producibility/*k*-entanglement II



$(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$ 2-entangled

$(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$ 3-entangled

$(|0000\rangle + |1111\rangle) \otimes (|0\rangle + |1\rangle)$ 4-entangled

Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z; $J_l = \sum_n j_l^{(n)}$.

- Symmetric Dicke states of spin-1/2 particles, with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

  Summing over all permutations.
  Due to symmetry, $\left\langle \vec{J}^2 \right\rangle$ is maximal.

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel. *et al.,* PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke, Science 2011; Hamley *et al*, Nat. Phys. 2012.

Spin Squeezing Inequality for Dicke states

- For separable states

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_z)^2$$

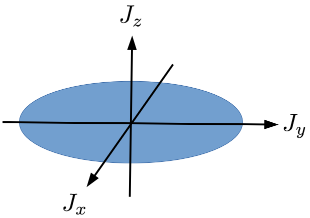
holds. GT, C. Knapp, O. Gühne, and H.J. Briegel, Phys. Rev. Lett. 2007

- It detects entangled states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \mathrm{max.},$$

$$\langle J_z^2 \rangle = 0.$$

- "Pancake" like uncertainty ellipse.



Multipartite entanglement - Dicke states

- Sørensen-Mølmer condition for k-producible states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right).$$

- Combine it with

$$\langle J_x^2 + J_y^2 \rangle \leqslant J_{\max}(\tfrac{k}{2} + 1) + \langle J_x \rangle^2 + \langle J_y \rangle^2,$$

which is true for pure k-producible states. (Remember, $J_{\max} = \frac{N}{2}$.)

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle - J_{\max}(\frac{k}{2} + 1)}}{J_{\max}} \right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states. [$F(x)$ is defined by Sørensen and Mølmer.]

G. Vitagliano *et al.*, NJP 2017.

Outline

1. Motivation

2. Multiparticle entanglement with collective observables
   - Theoretical background
   - Experiment in cold gases

3. Criteria with many-body correlations
   - Bipartite criterion
   - Multiparticle entanglement

Dicke state of cold atoms

PRL 112, 155304 (2014) PHYSICAL REVIEW LETTERS week ending 18 APRIL 2014

Detecting Multiparticle Entanglement of Dicke States

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Recent experiments demonstrate the production of many thousands of neutral atoms entangled in their spin degrees of freedom. We present a criterion for estimating the amount of entanglement based on a measurement of the global spin. It outperforms previous criteria and applies to a wider class of entangled states, including Dicke states. Experimentally, we produce a Dicke-like state using spin dynamics in a Bose-Einstein condensate. Our criterion proves that it contains at least genuine 28-particle entanglement. We infer a generalized squeezing parameter of $-11.4(5)$ dB.

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Entanglement, one of the most intriguing features of quantum mechanics, is nowadays a key ingredient for many applications in quantum information science [1,2], quantum simulation [3,4], and quantum-enhanced metrology [5]. Entangled states with a large number of particles cannot be characterized via full state tomography [6], which is routinely used in the case of photons [7,8], trapped ions [9], or superconducting circuits [10,11]. A reconstruction of the full density matrix is hindered and finally prevented by the exponential increase of the required number of measurements. Furthermore, it is technically impossible to address all individual particles or even fundamentally forbidden if the particles occupy the same quantum state. Therefore, the entanglement of many-particle states is best characterized by measuring the expectation values and variances of the components of the collective spin $\mathbf{J} = (J_x, J_y, J_z)^T = \sum_i \mathbf{s}_i$, the sum of all individual spins $\mathbf{s}_i$ in the ensemble.

In particular, the spin-squeezing parameter $\xi^2 = N(\Delta J_z)^2/(\langle J_x\rangle^2 + \langle J_y\rangle^2)$ defines the class of spin-squeezed states for $\xi^2 < 1$. This inequality can be used to verify the presence of entanglement, since all spin-squeezed states are entangled [12]. Large clouds of entangled neutral atoms are typically prepared in such spin-squeezed states, as shown in thermal gas cells [13], at ultracold temperatures [14–16], and in Bose-Einstein

quantified by means of the so-called entanglement depth, defined as the number of particles in the largest nonseparable subset [see Fig. 1(a)]. There have been numerous experiments detecting multiparticle entanglement involving up to 14 qubits in systems, where the particles can be addressed individually [9,20–24]. Large ensembles of neutral atoms

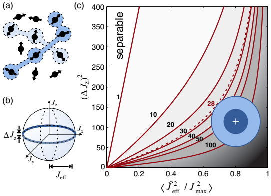


FIG. 1 (color online). Measurement of the entanglement depth for a total number of 8000 atoms. (a) The entanglement depth is given by the number of atoms in the largest nonseparable subset

Outline

Particle number resolving detection

- The resolution of the particle number detection is not 1 particle. Typically, ~ 10.
- So far we did not need single particle resolution.
- Particle-number resolving detection could improve the detected quality of the state dramatically.
- We could also have new entanglement criteria relying on single particle resolution.
- It is possible to reach a single-particle resolution:

  M. Quensen, M. Hetzel, L. Santos, A. Smerzi, G. Tóth, L. Pezzé, C. Klempt, Hong-Ou-Mandel interference of more than 10 indistinguishable atoms, arXiv:2504.02691.

Parity measurement

- We can measure the paritity as
$$\langle \sigma_z \otimes \sigma_z \otimes .... \otimes \sigma_z \rangle = \langle f(J_z) \rangle ,$$
where
$$f(z) = e^{i2\pi(z+N/2)}.$$
- E. g, for $N = 4$, we have
$$\{f(z)\}_{z=-2,-1,0,1,2} = \{+1, -1, +1, -1, +1\}.$$
- Thus, we do not need individual access to the particles, but we need a particle number resolving detection.

Dicke states and GHZ states

- Symmetric Dicke state with $\langle J_z \rangle = 0$

$$|D_N\rangle = \binom{N}{N/2}^{-1/2} \sum_k P_k(|0\rangle^{\otimes N/2}|1\rangle^{\otimes N/2}),$$

where P_k denote permutations different from each other.

- GHZ state

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}).$$

- For both states

$$|\langle \sigma_x \otimes \sigma_x \otimes ... \otimes \sigma_x \rangle| = |\langle \sigma_y \otimes \sigma_y \otimes ... \otimes \sigma_y \rangle| = |\langle \sigma_z \otimes \sigma_z \otimes ... \otimes \sigma_z \rangle| = 1$$

for $l = x, y, z$.

Entanglement conditions with many-body correlations

For separable states

$$|\langle\sigma_x\otimes\sigma_x\otimes\ldots\otimes\sigma_x\rangle|+|\langle\sigma_y\otimes\sigma_y\otimes\ldots\otimes\sigma_y\rangle|+|\langle\sigma_z\otimes\sigma_z\otimes\ldots\otimes\sigma_z\rangle|\leq 1$$

holds.

- For the ideal Dicke state the value is 3.

N	$\langle\sigma_x^{\otimes N}\rangle$	$\lvert\langle\sigma_z^{\otimes N}\rangle\rvert$	$\langle J_x^2+J_y^2\rangle$	$\mathcal{J}$	$(\Delta J_z)^2$
2	0.892(22)	0.965(13)	1.892(22)	0.946(11)	0.0176(66)
4	0.821(44)	0.951(25)	5.08(29)	0.85(5)	0.025(12)
6	0.833(61)	0.942(33)	11.26(85)	0.94(7)	0.029(17)
8	0.821(70)	0.806(70)	19.0(16)	0.95(8)	0.098(36)
10	0.872(72)	0.822(86)	25.7(26)	0.86(9)	0.091(45)
12	0.61(13)	0.862(96)	33.7(46)	0.80(11)	0.067(44)

Extended Data Table 1: Measurement results for various particle numbers. The uncertainties denote one standard deviation.

Proof

For separable states

$$\left|\langle\sigma_x \otimes \sigma_x \otimes ... \otimes \sigma_x\rangle\right| + \left|\langle\sigma_y \otimes \sigma_y \otimes ... \otimes \sigma_y\rangle\right| + \left|\langle\sigma_z \otimes \sigma_z \otimes ... \otimes \sigma_z\rangle\right| \leq 1$$

holds.

- *Proof.* For a product state of the type

  $$|\Psi^{(1)}\rangle \otimes |\Psi^{(2)}\rangle \otimes ... \otimes |\Psi^{(N)}\rangle$$

  the left-hand side can be bounded from above as

  $$\sum_{l=x,y,z}\left|\prod_{n=1}^{N}\left\langle\sigma_l^{(n)}\right\rangle\right| \leq \left|\left\langle\sigma_x^{(1)}\right\rangle\left\langle\sigma_x^{(2)}\right\rangle\right| + \left|\left\langle\sigma_y^{(1)}\right\rangle\left\langle\sigma_y^{(2)}\right\rangle\right| + \left|\left\langle\sigma_z^{(1)}\right\rangle\left\langle\sigma_z^{(2)}\right\rangle\right| \leq 1$$

  where in the first inequality we used that $\left|\left\langle\sigma_l^{(n)}\right\rangle\right| \leq 1$, and in the second inequality we used the Cauchy-Schwarz inequality and the fact that the length of the Bloch vector is at most one for a qubit.
- Separable states are mixtures of product states, hence the inequality is also valid for separable states. □

States detected

- The witness also detects the GHZ states as entangled.
- The singlet state given as

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \otimes \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \otimes ... \otimes \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

  has

$$(\Delta J_z)^2 = 0,$$

  and

$$\left\langle \sigma_x^{\otimes N} \right\rangle = 1, \quad \left\langle \sigma_y^{\otimes N} \right\rangle = 1, \quad \left\langle \sigma_z^{\otimes N} \right\rangle = 1,$$

  if N is divisible by 4.

- Thus, these operators cannot be used to detect genuine multipartite entanglement.

Inequality with multi-particle correlations

Observation 1. For N-qubit quantum states,

$$\langle J_x\rangle^2/j^2 + \langle J_y\rangle^2/j^2 + \langle \sigma_z^{\otimes N}\rangle^2 \leq 1$$

holds, where $j = N/2$ and

$$J_l = \frac{1}{2}\sum_{n=1}^{N} \sigma_l^{(n)}$$

for $l = x, y, z$.

Proof. The ground state of the Hamiltonian

$$H = BJ_x + K\sigma_z^{\otimes N},$$

where B and K are constants, is of the form

$$|\Psi\rangle = \alpha|0\rangle_x^{\otimes N} + \beta|1\rangle_x^{\otimes N},$$

which is a generalized GHZ state in the x-basis.

Inequality with multi-particle correlations II

Then, the relevant expectation value of J_x is

$$\langle J_x \rangle = \frac{N}{2} \langle \sigma_x \rangle_\phi$$

and the expectation value of the products of σ_z matrices is

$$\left\langle \sigma_z^{\otimes N} \right\rangle = \langle \sigma_z \rangle_\phi \,,$$

where we define the single-qubit state

$$|\phi\rangle = \alpha|0\rangle_x + \beta|1\rangle_x.$$

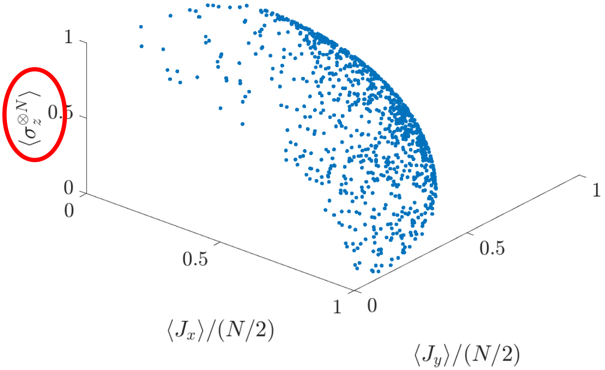
Since $\langle \sigma_x \rangle_\phi^2 + \langle \sigma_z \rangle_\phi^2 \leq 1$, it follows that

$$\langle J_x \rangle^2 / j^2 + \left\langle \sigma_z^{\otimes N} \right\rangle^2 \leq 1.$$

Then, assuming that the mean spin is not in the x-direction, but is in the xy-plane, we arrive at our inequality. □

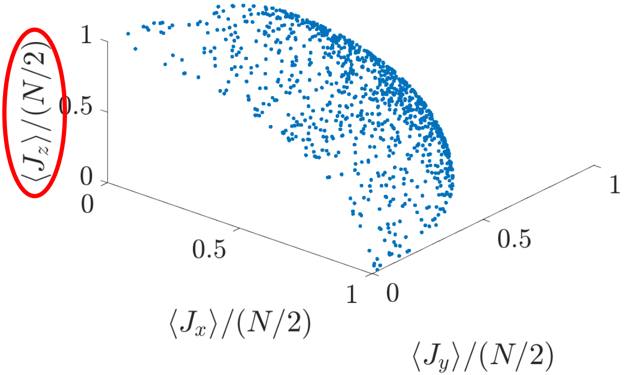
Inequality with multi-particle correlations III

Generalized GHZ states:



Inequality with multi-particle correlations IV

Comparison: spin coherent states



Bipartite conditions

Observation 2. For bipartite separable states,

$\sim \frac{1}{2}$ $\sim \frac{1}{2}$ 1 (maximal)←Dicke states

$$\langle J_x \otimes J_x \rangle/(j_1 j_2) + \langle J_y \otimes J_y \rangle/(j_1 j_2) + \left| \langle \sigma_z^{\otimes N_1} \otimes \sigma_z^{\otimes N_2} \rangle \right| \leq 1$$

holds, where for the left half we have

$$j_1 = N_1/2, \qquad j_2 = N_2/2.$$

N_1 particles	N_2 particles

Proof. We start from Observation 1

$$\langle J_x \rangle^2 / j^2 + \langle J_y \rangle^2 / j^2 + \langle \sigma_z^{\otimes N} \rangle^2 \leq 1$$

and use the Cauchy-Schwarz inequality. □

Bipartite conditions

- Problem: we need to measure observables in the two halves of the system.
- In many experiments, we measure only collective observables.
- We need to modify the inequality such that it works for that case.
- Note that we need to measure the particle number with a single particle resolution.

Bipartite conditions

Observation 3. The following expression is true for bipartite separable states

$$\left\langle J_x^2 + J_y^2 \right\rangle /(2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \leq j(j+1)/(2j_1 j_2),$$

where

$$j_1 = N_1/2, \qquad j_2 = N_2/2, \qquad j = N/2.$$

Proof. We start from the previous Observation. We add to both sides

$$\left\langle (J_x^{(1)})^2 + (J_y^{(1)})^2 \right\rangle /(2j_1 j_2) + \left\langle (J_x^{(2)})^2 + (J_y^{(2)})^2 \right\rangle /(2j_1 j_2).$$

Then follows the relation

$$\left\langle J_x^2 + J_y^2 \right\rangle /(2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right|$$
$$\leq 1 + \left\langle (J_x^{(1)})^2 + (J_y^{(1)})^2 \right\rangle /(2j_1 j_2) + \left\langle (J_x^{(2)})^2 + (J_y^{(2)})^2/(2j_1 j_2). \right\rangle$$

Bipartite conditions II

Then, starting from the relation

$$\left\langle J_x^2 + J_y^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \\ \leq 1 + \left\langle (J_x^{(1)})^2 + (J_y^{(1)})^2 \right\rangle / (2j_1 j_2) + \left\langle (J_x^{(2)})^2 + (J_y^{(2)})^2 \right\rangle / (2j_1 j_2),$$

we use the inequality

$$\left\langle (J_x^{(n)})^2 + (J_y^{(n)})^2 \right\rangle \leq j_n(j_n + 1).$$

We arrive at

$$\left\langle J_x^2 + J_y^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \leq j(j+1)/(2j_1 j_2).$$

We need to measure only collective quantities! □

Outline

Conditions for multi-particle entanglement

Observation 4. States violating the inequality

$$\left\langle J_x^2 + J_y^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \leq j(j+1)/(2j_1 j_2),$$

for

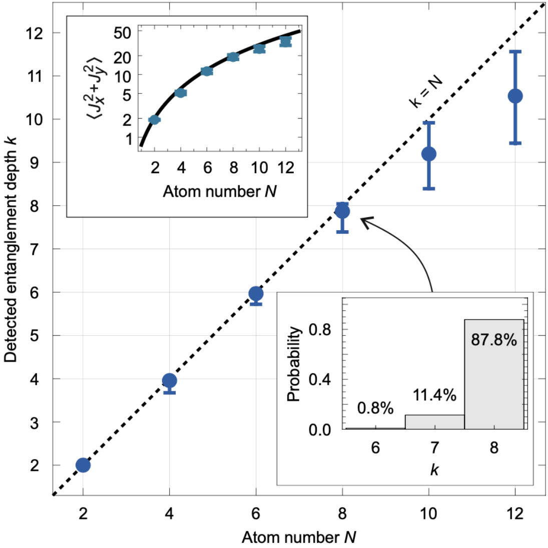
$$j_1 = k/2, \qquad j_2 = (N-k)/2$$

k particles	$N-k$ particles

possess at least $(k+1)$-particle entanglement, where we assume that $k \geq N/2$.

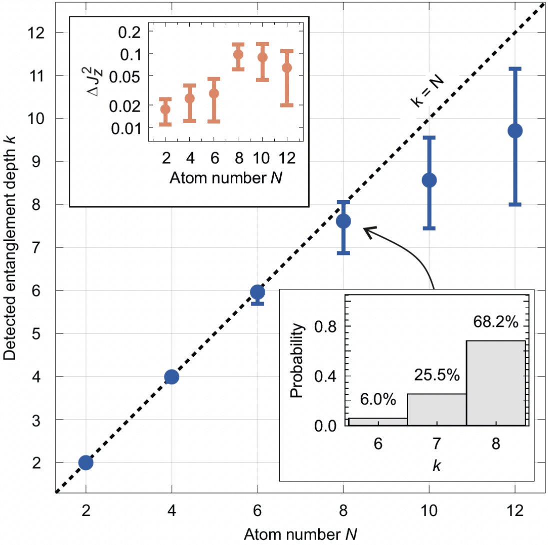
Violation for $k = N-1$ means genuine multipartite entanglement.

Results



Detected entanglement depth k
Atom number N
$k = N$
$\langle J_x^2 + J_y^2 \rangle$
Atom number N
Probability
0.8%
11.4%
87.8%
k

Comparison



Detected entanglement depth k
12
11
10
9
8
7
6
5
4
3
2
2
4
6
8
10
12
Atom number N
k = N
ΔJ_z^2
0.2
0.1
0.05
0.02
0.01
2
4
6
8
10
12
Atom number N
Probability
0.8
0.4
0.0
68.2%
25.5%
6.0%
6
7
8
k

Conclusions

- We discussed how to detect bipartite and multipartite entanglement with many-body correlation measurements.
- The method has been successfully used in experiments with Dicke states up to 12 particles.
- It demonstrates the good quality of the created Dicke state.
- For the transparencies, see

  www.gtoth.eu

- See also

M. Quensen, M. Hetzel, L. Santos, A. Smerzi,
G. Tóth, L. Pezzé, C. Klempt.

Hong-Ou-Mandel interference of more than 10 indistinguishable atoms,

Nature Physics (2026).

THANK YOU FOR YOUR ATTENTION!