Lower bounds on the quantum Fisher information based on the variance and various types of entropies

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Tromsø
Norwegian Sea
Kiruna
White Sea
Arkhangel
Reykjavík
ICELAND
Arctic Circle
Luleå
Oulu
NORWAY
Lake Onega
Umeå
FINLAND
Trondheim
SWEDEN
Gulf of Bothnia
Tampere
Lake Ladoga
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St. Petersburg
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Gävle
Gulf of Finland
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Tallinn
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Sevilla
Cagliari
Ionian Sea
Mediterranean Sea
Palermo
Gibraltar (U.K.)
Málaga
Sicily
Rhode
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Melilla (SPAIN)
Oran
Algiers
Tunis
Crete
Rabat
Casablanca
Valletta
MALTA
MOROCCO
ALGERIA
TUNISIA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles
Boundary representatio not necessarily authorit

Outline

1. **Motivation**
   - Why is the quantum Fisher information important?
2. **Bacground**
   - Quantum Fisher information
   - Recent findings on the quantum Fisher information
3. **Results**
   - Bounding the quantum Fisher information based on the variance

Why is the quantum Fisher information important?

- Many experiments are aiming to carry out a metrological task.
- If we can estimate the quantum Fisher information, we know how well this task could be carried out.
- Estimating the quantum Fisher information can be much simpler than carrying out the metrological task.

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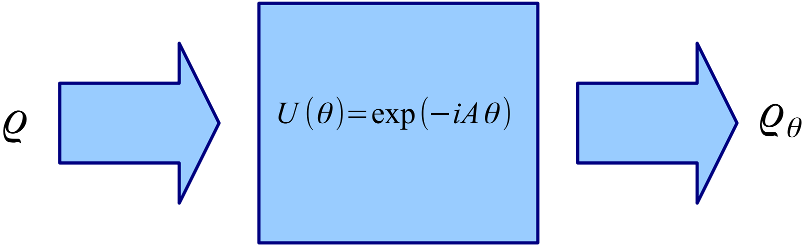
Quantum metrology

- Fundamental task in metrology



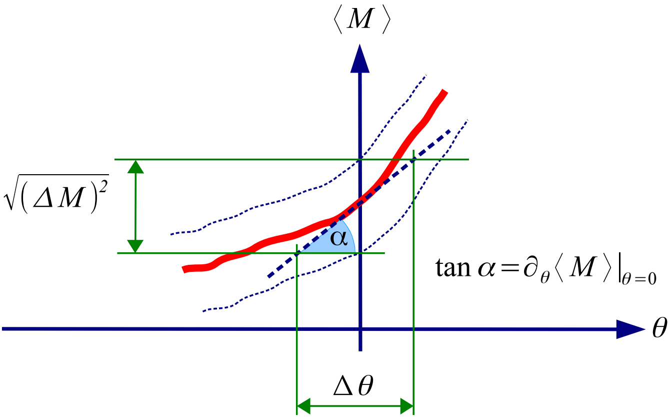
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

Precision of parameter estimation

- Measure an operator M to get the estimate θ. The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \qquad (\Delta\theta)^{-2} \leq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

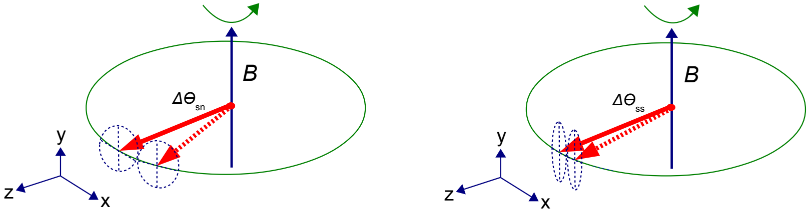
where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Special case $A = J_l$

- The operator A is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$

- Magnetometry with a linear interferometer



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Properties of the Fisher information

Many bounds on the quantum Fisher information can be derived from these simple properties:

- For pure states, it equals four times the variance, $F[|\Psi\rangle\langle\Psi|, A] = 4(\Delta A)^2{}_\Psi$.
- For mixed states, it is convex.

The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \leq N, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k-particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \leq kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Macroscopic superpositions (e.g, GHZ states, Dicke states)

$$F_Q[\varrho, J_l] \propto N^2,$$

[F. Fröwis, W. Dür, New J. Phys. 14 093039 (2012).]

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{p_k, \Psi_k} \sum_k p_k (\Delta A)^2{}_k,$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013); GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Thus, it is similar to entanglement measures that are also defined by convex roofs.

Witnessing the quantum Fisher information based on few measurements

- The bound based on $w = \mathrm{Tr}(\varrho W)$ is given as

$$F_Q[\varrho, J_z] \geq \sup_r \left[rw - \hat{\mathcal{F}}_{\mathrm{Q}}(rW) \right].$$

- The Legendre transform is

$$\hat{\mathcal{F}}_{\mathrm{Q}}(W) = \sup_{\varrho} (\langle W \rangle_{\varrho} - F_Q[\varrho, J_z]).$$

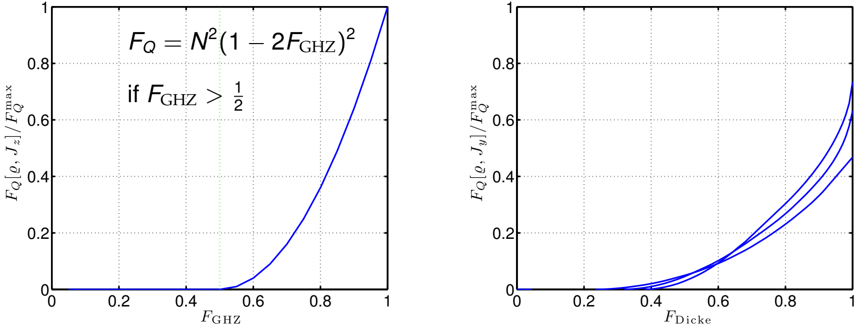
Due to the properties of F_Q mentioned above, it can be simplified

$$\hat{\mathcal{F}}_{\mathrm{Q}}(W) = \sup_{\mu} \left\{ \lambda_{\max} \left[W - 4(J_z - \mu)^2 \right] \right\}.$$

[I. Apellaniz, M. Kleinmann, O. Gühne, and G. Tóth, Phys. Rev. A 95, 032330 (2017), Editors' Suggestion.]

Example: bound based on fidelity

- Let us bound the quantum Fisher information based on some measurements.



Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for $N = 4, 6, 12$. $F_Q^{\rm max} = N^2$.

[Apellaniz *et al.*, Phys. Rev. A 2017]

Variance

- The variance is the concave roof of the itself

$$(\Delta A)^2_\varrho = \sup_{\{p_k,\Psi_k\}} \sum_k p_k (\Delta A)^2_{\Psi_k},$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013)]

- For 2×2 covariance matrices there is always $\{\Psi_k, p_k\}$ such that

$$C_\varrho = \sup_{\{p_k,\Psi_k\}} \sum_k p_k C_{\Psi_k},$$

[Z. Léka and D. Petz, Prob. and Math. Stat. 33, 191 (2013)]

- For 3×3 covariance matrices, this is not always possible. Necessary and sufficient conditions for an arbitrary dimension.
[D. Petz and D. Virosztek, Acta Sci. Math. (Szeged) 80, 681 (2014)]

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Bound based on the variance

- Let us define the quantity

$$V(\varrho, A) := (\Delta A)^2 - \frac{1}{4} F_Q[\varrho, A].$$

- It is well known that $V(\varrho, A) = 0$ for pure states.
- For states sufficiently pure $V(\varrho, A)$ is small.
- For states that are far from pure, the difference can be larger.

Generalized variance

- Generalized variances are defined as

$$\mathrm{var}_{\varrho}^{f}(A) \;=\; \sum_{ij} m_f(\lambda_i, \lambda_j)|A_{ij}|^2 - \Big(\sum \lambda_i A_{ii}\Big)^2,$$

  where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a matrix monotone function, and $m_f(a, b) = bf(b/a)$ is a corresponding mean.
  [Petz, J. Phys. A 35, 929 (2002); Gibilisco, Hiai, and Petz, IEEE Trans. Inf. Theory 55, 439 (2009)]

- We can define a large set of generalized variances, including for example the usual variance $\langle A^2\rangle - \langle A\rangle^2$.

- Consider $f_{\mathrm{har}} = 2x/(1+x)$. The corresponding mean is the harmonic mean $m_{f_{\mathrm{har}}}(a, b) = 2ab/(a+b)$. Direct calculations yield

$$\mathrm{var}_{\varrho}^{f_{\mathrm{har}}}(A) \equiv V(\varrho, A).$$

Bound based on the variance, rank-2

Observation 1.—For rank-2 states ϱ,

$$(\Delta A)^2 - \frac{1}{4}F_Q[\varrho, A] = \frac{1}{2}[1 - \mathrm{Tr}(\varrho^2)](\tilde{\sigma}_1 - \tilde{\sigma}_2)^2$$

holds, where $\tilde{\sigma}_k$ are the nonzero eigenvalues of the matrix

$$A_{kl} = \langle k|A|l\rangle.$$

Here $|k\rangle$ are the two eigenvectors of ϱ with nonzero eigenvalues. Thus, $\tilde{\sigma}_k$ are the eigenvalues of A on the range of ϱ.

Note

$$S_{\mathrm{lin}}(\varrho) = 1 - \mathrm{Tr}(\varrho^2) = 1 - \sum_k \lambda_k^2 = \sum_{k \neq l} \lambda_k \lambda_l.$$

Bound based on the variance, arbitrary rank

Observation 2.—For states ϱ with an arbitrary rank we have

$$(\Delta A)^2 - \frac{1}{4}F_Q[\varrho, A] \le 2S_{\rm lin}(\varrho)\sigma_{\max}(A^2),$$

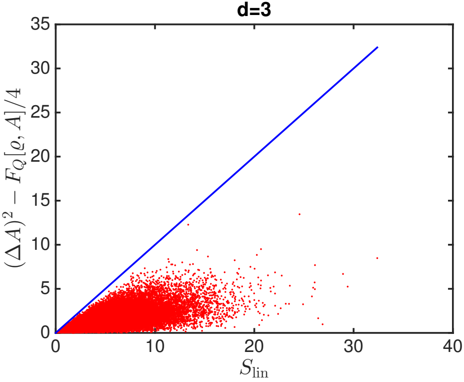
where $\sigma_{\max}(A^2)$ is the largest eigenvalue of A^2.

Estimate F_Q:

1. Measure the variance.
2. Estimate the purity.
3. Find a lower bound on F_Q.

Bound based on the variance, arbitrary rank II

Numerical verification of the bound



Quantities Averaged over SU(d) generators

- Traceless Hermitian matrices

$$A_{\vec{n}} := \vec{A}^T \vec{n},$$

where $\vec{A} = [A^{(1)}, A^{(2)}, A^{(3)}, ...]^T$ are the SU(d) generators.

- Average over unit vectors

$$\mathrm{avg}_{\vec{n}} f(\vec{n}) = \frac{\int f(\vec{n}) M(d\vec{n})}{\int M(d\vec{n})}.$$

- Compute average of V for operators.

- It is zero only for pure states. $\rightarrow$ Similar to entropies.

Bound on the average V

Observation 3.—The average of V over traceless Hermitian matrices with a fixed norm is given as

$$\underset{\substack{A: A=A^\dagger, \\ \mathrm{Tr}(A)=0, \\ \mathrm{Tr}(A^2)=2}}{\mathrm{avg}} V(\varrho, A) = \frac{2}{d^2-1}\left[S_{\mathrm{lin}}(\varrho) + H(\varrho) - 1\right],$$

where d is the dimension of the system, and

$$H(\varrho) = 2\sum_{k,l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l} = 1 + 2\sum_{k\neq l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l}.$$

Average quantum Fisher information

- The average of the quantum Fisher information can be obtained as
$$\mathrm{avg}_{\vec{n}} F_Q[\varrho, A_{\vec{n}}] = \frac{8}{N_{\mathrm{g}}}[d - H(\rho)].$$

- It is maximal for pure states.

Bound based on the variance II

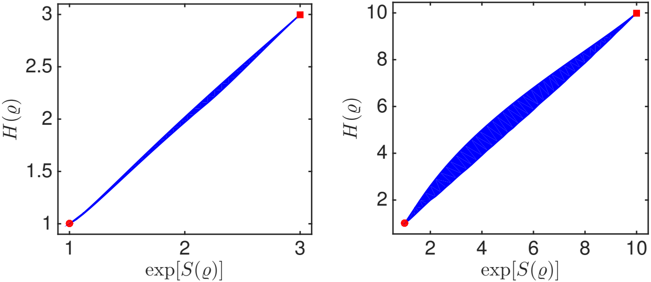


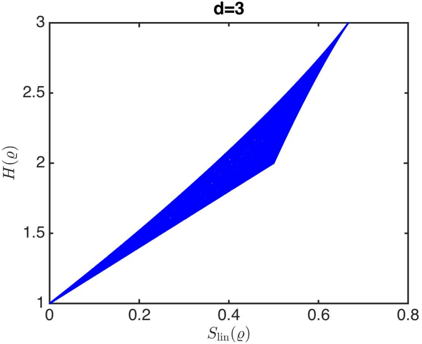
Figure: The relation between the von-Neumann entropy and $H(\varrho)$ for $d = 3$ and 10.
(filled area) Physical quantum states.
(dot) Pure states.
(square) Completely mixed state.

We see that

$$H(\varrho) \sim \exp[S(\varrho)].$$

What if we try the linear entropy

The relation between the two seems to be less strong.



Other type of quantum Fisher information

- The alternative form of the quantum Fisher information is defined as

$$
\begin{aligned}
F_Q(\varrho; A) &= 2\sum_{k,l} \frac{1}{\lambda_k + \lambda_l} |A_{kl}|^2 \\
&= \sum_k \frac{1}{\lambda_k} |A_{kk}|^2 + 2\sum_{k\neq l} \frac{1}{\lambda_k + \lambda_l} |A_{kl}|^2.
\end{aligned}
$$

- The quantum Fisher information defined above corresponds to estimating the parameter ϕ for the dynamics

$$\varrho_\phi = \varrho_0 + \phi A.$$

The Cramér-Rao bound in this case is

$$(\Delta\phi)^2 \geq \frac{1}{F_Q(\varrho; A)}.$$

Other type of quantum Fisher information

- In contrast, $F_Q[\varrho, A]$ corresponds to estimating the parameter θ of the unitary evolution

$$\varrho_\theta = \exp(-iJ_z\theta)\varrho_0 \exp(+iJ_z\theta),$$

  as discussed in the introduction.

- The relation of the two types of quantum Fisher information is given by

$$F_Q[\varrho, A] = F_Q(\varrho; i[\varrho, A]).$$

Other type of quantum Fisher information II

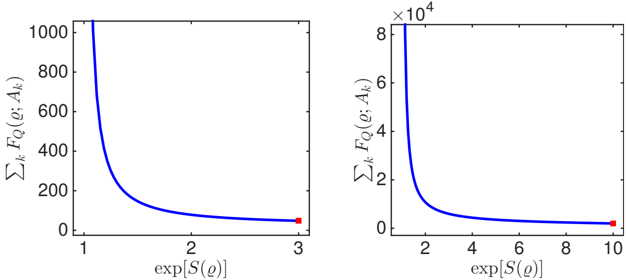


Figure: The relation between the von-Neumann entropy $S(\varrho)$ and the average $F(\varrho; A)$ defined in for $d = 3$ and 10.

(solid) Points corresponding to the states giving the minimum (pure state mixed with white noise).

(square) Completely mixed state.

Generalized quantum Fisher information (D. Petz)

- The gerealized QFI is defined as

$$F_Q^f(\varrho; A) = \sum_{k,l} \frac{1}{m_f(\lambda_k, \lambda_l)} |A_{kl}|^2, \quad (1)$$

  where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a matrix monotone function, and $m_f(a, b) = bf(b/a)$ is a corresponding mean.

- Similarly, as before we can define

$$F_Q^f[\varrho, A] = F_Q^f(\varrho; i[\varrho, A]).$$

Kubo-Mori-Bogoliubov quantum Fisher information

- Let us consider $f_{\log}(x) = (x-1)/\ln x$, which corresponds to the logarithmic mean

$$m_{f_{\log}}(a,b) = \frac{a-b}{\ln a - \ln b}.$$

- The corresponding genetralized quantum Fisher information is defined as

$$\begin{aligned} F_Q^{\log}(\varrho; A) &= \sum_{k,l} \frac{\log(\lambda_k) - \log(\lambda_l)}{\lambda_k - \lambda_l} |A_{kl}|^2 \\ &= \sum_{k \neq l} \frac{\log(\lambda_k) - \log(\lambda_l)}{\lambda_k - \lambda_l} |A_{kl}|^2 + \sum_k \frac{1}{\lambda_k}. \end{aligned}$$

Kubo-Mori-Bogoliubov quantum Fisher information II

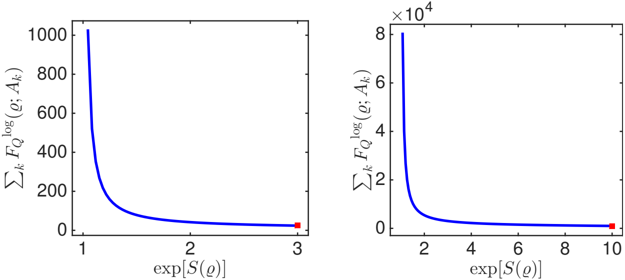


Figure: The relation between the von-Neumann entropy $S(\varrho)$ and the average $F^{\log}(\varrho; A)$ for $d = 3$ and 10.

(solid) Points corresponding to the states giving the minimum (pure state mixed with white noise).

(square) Completely mixed state.

Kubo-Mori-Bogoliubov quantum Fisher information III

Let us calculate

$$F_Q^{\log}[\varrho, A] = F_Q^{\log}(\varrho; i[\varrho, A]).$$

We obtain

$$F_Q^{\log}[\varrho, A] = \sum_{k,l} [\log(\lambda_k) - \log(\lambda_l)](\lambda_k - \lambda_l)|A_{kl}|^2.$$

With this one can show that

$$\frac{d^2}{d^2\theta} S(\varrho || e^{-iA\theta} \varrho e^{+iA\theta})|_{\theta=0} = F_Q^{\log}[\varrho, A],$$

and

$$\mathrm{avg}_{\vec{n}} F_Q^{\log}[\varrho, A_{\vec{n}}] = -\frac{2}{N_\mathrm{g}}(2dS + 2\sum_k \log \lambda_k).$$

We get a similar curve for the minimum.

Kubo-Mori-Bogoliubov quantum Fisher information IV

- Relation to other works in the literature.

- S. Huber, R. Koenig, and A. Vershynina, Geometric inequalities from phase space translations, arxiv:1606.08603.

  They establish a quantum version of the classical isoperimetric inequality relating the Fisher information and the entropy power of a quantum state.

- C. Rouze, N. Datta, and Y. Pautrat, Contractivity properties of a quantum diffusion semigrou, arxiv:1607.04242.

Summary

- We discussed how to find lower bounds on the quantum Fisher information and entropies.

See:
G. Tóth,

Lower bounds on the quantum Fisher information based on the variance and various types of entropies, arxiv:1701.07461.

THANK YOU FOR YOUR ATTENTION!

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