Coherent Sequences of Measurements and a Triple-Slit Interference

(with 3 figures)

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INTRODUCTION

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A theory-independent notion of coherent sequences of measurements.

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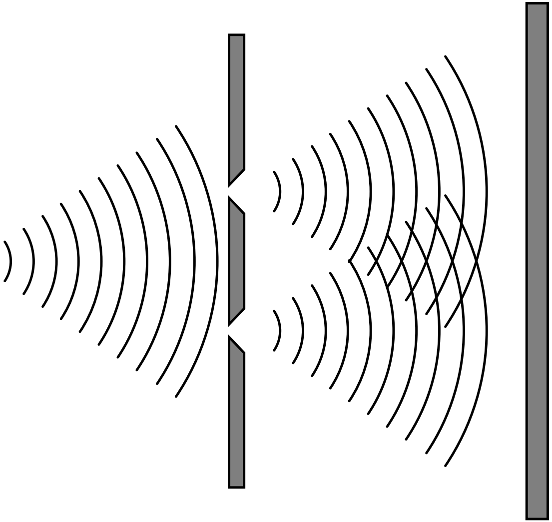
Why?

Text book example: The double-slit experiment



Text book example: The double-slit experiment

Superposition principle

$$\Psi(\mathbf{x},t) = \psi_1(\mathbf{x},t) + \psi_2(\mathbf{x},t)$$

Schrödinger equation

$$i\partial_t \psi_j = -\frac{1}{2m}\partial_{\mathbf{x}}^2 \psi_j \quad \hookrightarrow \quad \psi_j(\mathbf{x},t) = \int \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{r}_j - t\mathbf{k}/2m)}\Phi(\mathbf{k})\mathrm{d}\mathbf{k}$$

Born rule

$$\begin{aligned} I_{12}(\mathbf{x}) &= |\Psi(\mathbf{x},t)|^2 \\ &= |\psi_1(\mathbf{x},t)|^2 + |\psi_2(\mathbf{x},t)|^2 + 2\,\mathrm{Re}[\psi_1(\mathbf{x},t)^*\psi_2(\mathbf{x},t)] \\ &= I_1(\mathbf{x}) + I_2(\mathbf{x}) + \mathfrak{I}_{12}(\mathbf{x}) \end{aligned}$$

Pure double-slit correlations

There is a measurement (Π_1, Π_2), such that

$$|\psi_k\rangle = \Pi_k|\Psi\rangle.$$

Write $\Delta_{\mathbf{x}}$ for detection at $\mathbf{x}$. Then

- single-slit: $I_k(\mathbf{x}) = \langle\psi_k|\Delta_{\mathbf{x}}|\psi_k\rangle = \langle\Psi|\Pi_k\Delta_{\mathbf{x}}\Pi_k|\Psi\rangle$
- double-slit: $I_{12}(\mathbf{x}) = \langle\Psi|\Delta_{\mathbf{x}}|\Psi\rangle$

Double-slit interference

With $\phi_k\colon A \mapsto \Pi_k A \Pi_k$ we have

$$\mathfrak{I}_{12}(\mathbf{x}) \equiv I_{12}(\mathbf{x}) - I_1(\mathbf{x}) - I_2(\mathbf{x}) = \langle\Psi|(\mathrm{id} - \phi_1 - \phi_2)[\Delta_{\mathbf{x}}]|\Psi\rangle$$

When double-slit correlations are universal

n-slit measurement

For $\alpha \subset \{1, \ldots, n\}$ let

$$\Pi_\alpha = \sum_{j\in\alpha} \Pi_j \quad \text{and} \quad \phi_\alpha[A] = \Pi_\alpha A \Pi_\alpha.$$

Then: $\Big(\phi_\alpha - \sum_{j\in\alpha} \phi_{\{j\}}\Big)[A] = \sum_{i<j;\; i,j\in\alpha} \Big(\phi_{\{i,j\}} - \phi_{\{i\}} - \phi_{\{j\}}\Big)[A]$

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Fact [Sorkin, Mod. Phys. Lett. (1995); Sinha et al., Science (2007)]

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Example (triple-slit correlations)

Choose $F_1 = \mathbb{1}/2$, $F_2 = |0\rangle\langle 0|/2$, and $F_3 = |1\rangle\langle 1|/2$.

The existence of projective measurements: Lüders' rule

Lüders' rule [Ann. Phys., 1951]

For $A = \sum_a a\Pi_a$ there exists an implementation, so that

$$\begin{aligned} \langle B|A=a\rangle &= \mathrm{tr}(\varrho\,\Pi_a B\Pi_a)/\,\mathrm{tr}(\varrho\,\Pi_a) \\ &= \mathrm{tr}[\varrho\,\phi_a(B)]/\,\mathrm{tr}[\varrho\,\phi_a(\mathbb{1})], \end{aligned}$$

with $\phi_a(B) = \Pi_a B\Pi_a$.

Differs from suggestion by von Neumann, where

$$\phi_a(B) = \sum_k |\phi_k^a\rangle\langle\phi_k^a|B|\phi_k^a\rangle\langle\phi_k^a|$$

with $(|\phi_k^a\rangle)_k$ is an orthonormal basis of $\mathrm{range}\,\Pi_a$.

Is Lüders' rule an axiom?

Ozawas theorem [JMP, 1984]

For a collection of quantum maps ϕ_a with $\sum_a \phi_a(\mathbb{1}) = \mathbb{1}$ there exists $|\mathrm{anc}\rangle$, U, and an orthonormal basis $(|a\rangle_{\mathrm{anc}})$, so that

$$\mathrm{tr}(\varrho\,\phi_a[B]) = \mathrm{tr}[(|\mathrm{anc}\rangle\langle\mathrm{anc}| \otimes \varrho)U(|a\rangle\langle a| \otimes B)U^\dagger].$$

for all B.

$\hookrightarrow$ Existence of Lüders' rule is guaranteed by existence of non-local unitaries (and pure state preparation).

Use of a general definition of Lüders' rule

- When is a measurement projective?
- What makes a map ϕ_k a Lüders' rule?
- How to describe sequential measurements without enrolling Hilbert space formalism?

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Concepts premising projective sequential measurements:

- no triple-slit correlations
- macro-realism
- contextuality
- exclusivity principle (orthomodularity)

F O R M A L I S M

Quantum mechanics and positivity (i)

What do we need for quantum mechanics?

Measurements. A general measurement is a POVM,

$$(E_1, E_2, \dots) \text{ obeying } E_k \geq 0 \text{ and } \sum_k E_k = \mathbb{1}.$$

States. A general state is an ensemble ϱ or a map

$$\omega\colon A \mapsto \mathrm{tr}(\varrho A) \text{ so that } \omega(E_k) = p_k.$$

In particular $\omega(E) \geq 0$ for any $E \geq 0$ and $\omega(\mathbb{1}) = 1$.

Channels. A linear map ϕ is a quantum channel, if

$$\phi(E) \geq 0 \text{ for all } E \geq 0, \text{ and } (\mathrm{id} \otimes \phi)(E_{\mathrm{AB}}) \geq 0 \text{ for all extensions.}$$

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Order unit vector space

An order unit vector space is a triple $(V, \geq, e)$ obeying

- V is a real vector space.
- $\geq$ is a partial order on V with
  - $f + v \geq g + v$ if $f \geq g$
  - $\lambda f \geq 0$ if $f \geq 0$ and $\lambda \in \mathbb{R}_0^+$
- e is an order unit, i.e., for any $v \in V$ there exists a $\lambda > 0$ with $\lambda e \geq v$.

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The set of states is

$$\mathcal{S} = \{\omega \in V^* \mid \omega(f) \geq 0 \text{ for all } f \geq 0, \quad \omega(e) = 1\}.$$

Generalized probabilistic theories

How does an order unit vector space define a theory?

Measurements. A family $(f_1, f_2, \dots)$ is a measurement only if $f_k \geq 0$ and $\sum_k f_k = e$.

States. A state $\omega \in \mathcal{S}$ yields probabilities $\omega(f_k) = p_k$.

Channels. $\phi\colon V \to V$ is a channel only if

$$\phi(f) \geq 0 \text{ for all } f \geq 0 \text{ (positivity)}$$

DEFINITION

Coherent Lüders' rule for order unit vector spaces

Definition

A positive linear map $f^\sharp$ is a Lüders' rule for $0 \leq f \leq e$ if it is

- f-compatible, i.e., $f^\sharp e = f$, and
- repeatable, $f^\sharp f = f$.

Example (Failure!)

The von-Neumann measurements are repeatable.

Coherent Lüders' rule for order unit vector spaces

Definition (CLR)

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$$E^\sharp(B) = EBE.$$

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- The effects 0, e, and all extremal rays admit a CLR:
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  - f extremal: $f^\sharp(x) = f\omega(x)$ with $\omega(f) = 1$.

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  - $e^\sharp(x) = x$,
  - f extremal: $f^\sharp(x) = f\omega(x)$ with $\omega(f) = 1$.
- In general, a CLR is not unique.

More properties of CLRs

Repeatability, exclusivity, and compatibility

- $f^\sharp \circ f^\sharp = f^\sharp$.
- If $f + g \leq e$, then $f^\sharp \circ g^\sharp = 0$.
- If $g \leq f$, then $f^\sharp(g) = g^\sharp(f)$

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Assume some appropriate cone $\mathcal{C}$ of positive maps.

Theorem

An f-compatible positive map ϕ is a CLR for f if and only if $\phi \circ \psi = \psi$ holds for all f-compatible maps $\psi \in \mathcal{C}$.

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Robustness under sections:
If $\tau \colon W \to V$ with $W \subset V$, then $\tau^{-1} \circ \tau(f)^\sharp \circ \tau$ is a CLR for $\tau(f)$.

Alternative: Filters

Definition (Araki, CMP, 1980)

A positive linear map $f^\natural$ is a filter for $0 \leq f \leq e$ if it is

- f-compatible
- projective, i.e., $f^\natural \circ f^\natural = f^\natural$, and
- neutral, $\omega \circ f^\natural = \omega$ if $\omega(f) = 1$.

See also Alfsen & Shultz (since 1980'ies), Niestegge (since 2008), Ududec & Barnum & Emerson (2011).

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Objections:

- Origin in theory of propositions ($\hookrightarrow$ Gleason's theorem).
- Robustness under sections unclear.
- Some elements from the extreme boundary may not admit a filter (e.g. for Popescu-Rohrlich boxes).
- Spekkens toy model has a CLR.

Example: Dichotomic norm cones

Generalized Bloch sphere

Assume $V = \mathbb{R} \times \mathbb{R}^d$, $(t, \mathbf{x}) \geq 0$ if $t \geq \|\mathbf{x}\|$, and $e = (1, \mathbf{0})$.
Then only 0, e, and the extremal elements admit a CLR.

Embraces classical bit, qubit, hyperbits, gbit, and Spekkens toy model.

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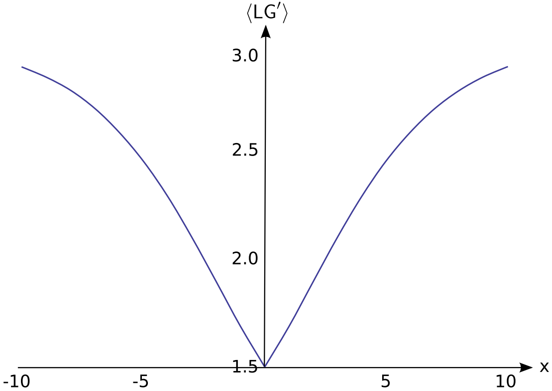
Simplified Leggett-Garg inequality

$$\langle \mathsf{LG}' \rangle = \langle AB \rangle_{\mathrm{seq}} + \langle B \rangle - \langle A \rangle$$

- macro-realistic bound $\langle \mathsf{LG}' \rangle \leq 1$
- quantum bound $\langle \mathsf{LG}' \rangle \leq 3/2$
- "algebraic" bound $\langle \mathsf{LG}' \rangle \leq 3$

Example: Dichotomic norm cones

$$\langle \mathsf{LG}' \rangle \leq \|\mathbf{a} - \mathbf{b}\| + \mathbf{a}' \cdot \mathbf{b}$$
$$\text{with } \|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{a}'\|_* = \mathbf{a}' \cdot \mathbf{a} = 1.$$

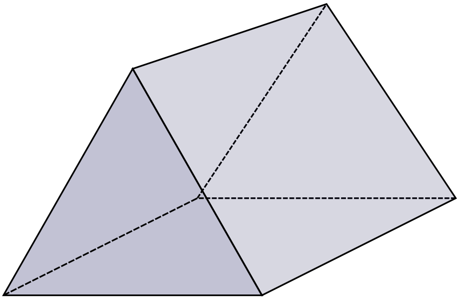


$$\|\mathbf{a}\| \equiv \|\mathbf{a}\|_q;\ q = 1.5^{\mathsf{x}} + 1;\ d = 2$$

Example: Triple-slit correlations

There exists an order unit vector space with maps ϕ_α, $\alpha \subset \{1,2,3\}$, such that

- $\phi_{\{k,j\}} - \phi_{\{k\}} - \phi_{\{j\}} = 0$, for all $k < j$,
- but $\phi_{\{1,2,3\}} - \sum_k \phi_{\{k\}} \neq 0$.



set of states

Are the elements with CLR projections?

In quantum mechanics each of the following requires f to be a projection:

(i) f admits a CLR.

(ii) $g \leq f$ implies $g \leq f\|g\|$

(iii) $g \leq f \leq e - g$ only for $g = 0$

(iv) $f = \sum g_k$ with g_k extremal

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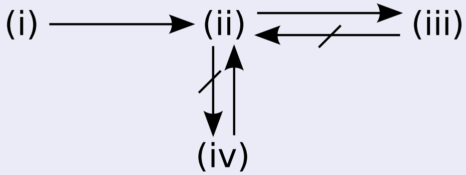
Proposition



SUMMARY

- Fundamental quantum features are based on coherent sequential measurements.
- Incoherent sequential quantum measurements may cause spurious post-quantum effects.
- Axiomatic description via coherent Lüders' rule or via filters.
- Sequential measurements in toy-theories exhibit post-quantum effects.

$\hookrightarrow$ arXiv:1402.3583

- related work by Chiribella & Yuan, arXiv:1404.3348.