Precision Bounds for Gradient Magnetometry

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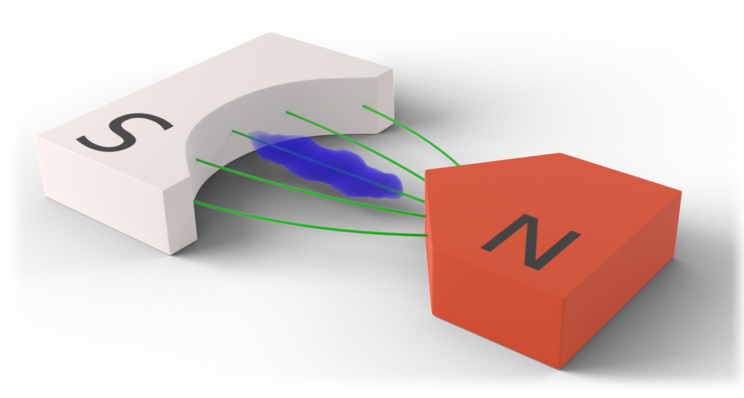
Abstract. We study gradient magnetometry for various quantum states. Based on the quantum Fisher information, we calculate precision bounds for estimating the gradient of the magnetic field. We consider a 1-dimensional lattice of atoms, two separated ensembles, a single atomic cloud and a Bose-Einstein condensate to model the spatial parts of the systems. For quantum states sensitive to the homogeneous magnetic field, a simultaneous estimation of the gradient and the homogeneous field is needed in order saturate our bounds. In all the cases we present in this work, we domonstrate that the precision bounds are saturable, and in some cases they can even reach the Heisenberg scaling.

The setup



Representation of an atomic clound in a magnetic field with a gradient term different from zero, similar to the Stern-Gerlach apparatus.

- For 1D atomic systems, the **magnetic field** can be represented by a linear equation as

$$\boldsymbol{B}(x,0,0)=\boldsymbol{B}_0+x\boldsymbol{B}_1+\mathcal{O}(x^2).$$

- For N point-like particles, we assume that **the state** is a product of the internal and spatial parts,

$$\varrho=\varrho^{(\mathrm{x})}\otimes\varrho^{(\mathrm{s})}\,,$$

$$\varrho^{(\mathrm{x})}=\int\frac{P(\boldsymbol{x})}{\langle\boldsymbol{x}|\boldsymbol{x}\rangle}|\boldsymbol{x}\rangle\langle\boldsymbol{x}|\,\mathrm{d}\boldsymbol{x}\,,$$

where $\boldsymbol{x}=(x_1,x_2,\dots,x_N)$.

- The **Interaction** Hamiltonian between the magnetic field and a single particle is

$$h^{(n)}=\gamma B_z^{(n)}\otimes j_z^{(n)}\,,$$

where $\gamma=g\mu_{\mathrm{B}}$.

- Colectivelly, the **unitary** evolution operator can be written as

$$U=e^{-i(b_0H_0+b_1H_1)}\,,$$

where

$$H_0=\sum_{n=1}^{N}j_z^{(n)}=J_z\,,$$

$$H_1=\sum_{n=1}^{N}x^{(n)}j_z^{(n)}.$$

In this work, we find bounds for b_1, even when the state is sensitive to the homogeneous fields. In the last case a simultaneous measurement is needed to saturate the bound.

Cramér-Rao bounds

When computing the CR bounds one must take into account that the homogeneous field may affect the state.

The **quantum Fisher information** for two arbitrary operators is

$$\mathcal{F}_{\mathrm{Q}}[\varrho,A,B]:=2\sum_{k,k'}\frac{(p_k-p_{k'})^2}{p_k+p_{k'}}A_{k,k'}B_{k',k}\,.$$

For separable states the CR cannot surpass the **shot-noise scaling**, ~N. The ultimate limit is called **Heisenberg scaling**, ~N^2.

Bound for states insensitive to the homogeneous fields

For states **insensitive** to the homogeneous fields, we can use the CR bound for a single parameter,

$$(\Delta b_1)^{-2}|_{\max}=\mathcal{F}_{\mathrm{Q}}[\varrho,H_1,H_1]=\mathcal{F}_{\mathrm{Q}}[\varrho,H_1]\,.$$

For point-like particles the **matrix elements** of the Hamiltonian are

$$(H_1)_{\boldsymbol{x},\lambda;\boldsymbol{y},\nu}=\delta(\boldsymbol{x}-\boldsymbol{y})\langle\lambda|\sum_{n=1}^{N}x_nj^{(n)}|\nu\rangle\,.$$

Finally, we arrive at

$$(\Delta b_1)^{-2}|_{\max}=\sum_{n,m}^{N}\int x_nx_mP(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})},j_z^{(n)},j_z^{(m)}]\,.$$

The bound is **invariant** under translations of the system, represented by

$$U_d=\exp(-idP_x)\,,$$

where P_x is the collective operator of the linear momentum along x and d is the displaced distance.

Bound for states sensitive to the homogeneous fields

For states **sensitive** to the homogeneous fields a second unkown parameter has to be considered, b_0. The CR bound becomes into a matrix inequality,

$$\boldsymbol{C}\geq\boldsymbol{\mathcal{F}}_{\mathrm{Q}}^{-1},$$

where $C_{ij}=\langle b_ib_j\rangle-\langle b_i\rangle\langle b_j\rangle$ and $\mathcal{F}_{ij}:=\mathcal{F}_{\mathrm{Q}}[\varrho,H_i,H_j]$.

The bound for the precision of the gradient estimation is

$$(\Delta b_1)^{-2}\leqslant\mathcal{F}_{11}-\frac{\mathcal{F}_{01}\mathcal{F}_{10}}{\mathcal{F}_{00}}\,.$$

We have that for point-like particles, the **matrix elements** of the operator related to the homogeneous field is

$$(H_0)_{\boldsymbol{x},\lambda;\boldsymbol{y},\nu}=\delta(\boldsymbol{x}-\boldsymbol{y})\langle\lambda|\sum_{n=1}^{N}j_z^{(n)}|\nu\rangle,$$

and hence,

$$\mathcal{F}_{11}=\sum_{n,m}^{N}\int x_nx_mP(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})},j_z^{(n)},j_z^{(m)}]\,,$$

$$\mathcal{F}_{01}=\sum_{n=1}^{N}\int x_nP(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})},j_z^{(n)},J_z]\,,$$

$$\mathcal{F}_{00}=\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})},J_z]\,.$$

Finally, for point-like particles, the precision bound is

$$(\Delta b_1)^{-2}\leqslant\sum_{n,m}^{N}\int x_nx_mP(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})},j_z^{(n)},j_z^{(m)}]-\frac{\left(\sum_{n=1}^{N}\int x_nP(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})},j_z^{(n)},J_z]\right)^2}{\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})},J_z]}\,.$$

The bound is **invariant** under translations of the system.

Measurements compatibility

The symmetric logarithmic derivative for an arbitrary operator is

$$L(\varrho,A)=2i\sum_{\lambda\neq\nu}\frac{p_\lambda-p_\nu}{p_\lambda+p_\nu}\langle\lambda|A|\nu\rangle|\lambda\rangle\langle\nu|\,.$$

The condition for a simultaneous measurement which allows to optimally estimate the gradient parameter is

$$[L(\varrho,H_0),L(\varrho,H_1)]=0\,.$$

For point-like particles we have that

$$L(\varrho,H_0)=\mathbf{1}^{(\mathrm{x})}\otimes L(\varrho^{(s)},J_z)$$

$$L(\varrho,H_1)=\sum_{n=1}^{N}\int\mathrm{d}\boldsymbol{x}\,x_n|\boldsymbol{x}\rangle\langle\boldsymbol{x}|\otimes L(\varrho^{(s)},j_z^{(n)})\,.$$

Finally, for permutationally invariant spin states

$$L(\varrho,H_1)=\hat{\mu}^{(\mathrm{x})}\otimes L(\varrho^{(s)},J_z)\,,$$

and for two different permutationally invariant ensembles and product spin states,

$$L(\varrho,H_1)=\hat{\mu}^{(\mathrm{L})}\otimes\mathbf{1}^{(\mathrm{R,x})}\otimes L(|\psi\rangle^{(\mathrm{L})},J_z^{(\mathrm{L})})\otimes\Pi_{|\psi\rangle}^{(\mathrm{R})}+\mathbf{1}^{(\mathrm{L,x})}\otimes\hat{\mu}^{(\mathrm{R})}\otimes\Pi_{|\psi\rangle}^{(\mathrm{L})}\otimes L(|\psi\rangle^{(\mathrm{R})},J_z^{(\mathrm{R})}),$$

in which, in both cases, the condition for a simutalneous measurement holds.

Some interesting properties of the $\mathcal{F}_{\mathrm{Q}}[\varrho, A, B]$ are:

- *Linear* $\quad\mathcal{F}_{\mathrm{Q}}[\varrho,\sum_i A_i,\sum_j B_j]=\sum_{i,j}\mathcal{F}_{\mathrm{Q}}[\varrho,A_i,B_j]$
- *A <-> B* $\quad\mathcal{F}_{\mathrm{Q}}[\varrho,A,B]=\mathcal{F}_{\mathrm{Q}}[\varrho,B,A]$
- *Pure st.* $\quad\mathcal{F}_{\mathrm{Q}}[|\psi\rangle,A,B]=4\left(\langle AB\rangle_\psi-\langle A\rangle_\psi\langle B\rangle_\psi\right)$
- *Convex* $\quad\mathcal{F}_{\mathrm{Q}}[p\varrho_1+(1-p)\varrho_2,A]\leqslant p\mathcal{F}_{\mathrm{Q}}[\varrho_1,A]+(1-p)\mathcal{F}_{\mathrm{Q}}[\varrho_2,A]$

We introduce important quantities, the average position of all particles, the variance, and the correlation among the position of the particles.

We are interested in the scaling of the bounds as a function of the particle number.

Introducing more particles into the system could increase improve the precision, merely because its size increases.

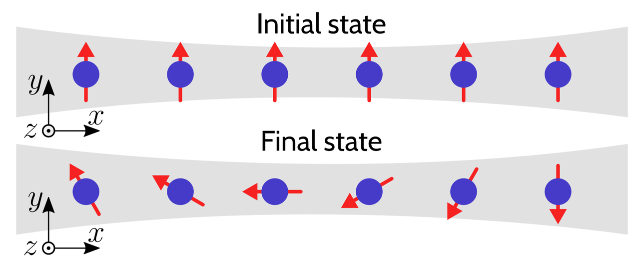
$$\mu=\int\frac{\sum_{n=1}^{N}x_n}{N}P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

$$\sigma^2=\int\frac{\sum_{n=1}^{N}x_n^2}{N}P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}-\mu^2$$

$$\eta=\int\frac{\sum_{n\neq m}^{N}x_nx_m}{N(N-1)}P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}-\mu^2$$

Spin-chain and double well

A **spin-chain** under the magnetic field. Initially all the spins point to the same direction. After the interaction the collective spin is decreased.



For the **spin-chain**, we have the following probability distribution function, and hence the variance which characterizes the spatial state,

$$P(\boldsymbol{x})=\prod_{n=1}^{N}\delta(x_n-na)\quad\text{and}\quad\sigma_{\mathrm{ch}}^2=a^2\frac{N^2-1}{12}\,.$$

Hence for the totally polarized state, $|\psi_{\mathrm{tp}}\rangle=|j\rangle_y^{\otimes N}$, the achievable precision is

$$(\Delta b_1)^{-2}|_{\max}=2\sigma_{\mathrm{ch}}^2Nj\,.$$

Similarly, for the **double well**, we have

$$P(\boldsymbol{x})=\prod_{n=1}^{N/2}\delta(x_n+a)\prod_{n=N/2+1}^{N}\delta(x_n-a)\quad\text{and}\quad\sigma_{\mathrm{dw}}^2=a^2.$$

The optimal state which maximizes the variance of H_1 is

$$|\psi\rangle=\frac{|j\cdots j\rangle^{(\mathrm{L})}|-j\cdots-j\rangle^{(\mathrm{R})}+|-j\cdots-j\rangle^{(\mathrm{L})}|j\cdots j\rangle^{(\mathrm{R})}}{\sqrt{2}}\,,$$

with which we obtain the best achievable precision for a given size, σ^2,

$$(\Delta b_1)^{-2}|_{\max}=4\sigma_{\mathrm{dw}}^2N^2j^2.$$

General bound for internal product states in the double-well

Interestingly for product states, $|\psi\rangle^{(\mathrm{L})}\otimes|\psi\rangle^{(\mathrm{R})}$, the achievable precision is

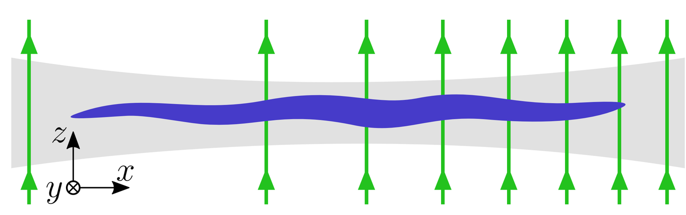
$$\mathcal{F}_{\mathrm{Q}}[|\psi\rangle^{(\mathrm{L})}|\psi\rangle^{(\mathrm{R})},H_1]=2a^2\mathcal{F}_{\mathrm{Q}}[|\psi\rangle^{(\mathrm{L})},J_z^{(\mathrm{L})}].$$

Hence, we can write the precision bounds for the gradient as a function of the precision the state has when estimating the homogeneous field.

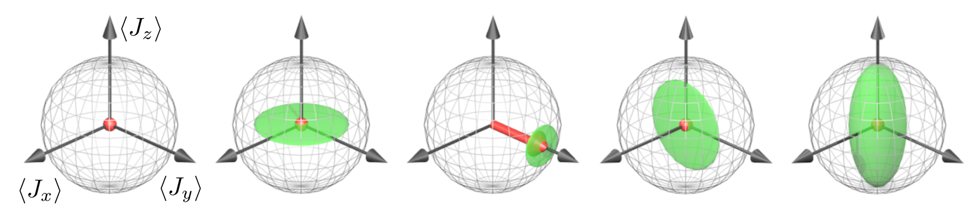
States	$\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{L})},J_z^{(\mathrm{L})}]$	$(\Delta b_1)^{-2}\lvert_{\max}$
$\lvert j\rangle_y^{\otimes N_{\mathrm{L}}}\otimes\lvert j\rangle_y^{\otimes N_{\mathrm{L}}}$	$2N_{\mathrm{L}}j$	$2a^2Nj$
$\lvert\Psi_{\mathrm{sep}}\rangle\otimes\lvert\Psi_{\mathrm{sep}}\rangle$	$4N_{\mathrm{L}}j^2$	$4a^2Nj^2$
$\lvert\mathrm{GHZ}\rangle\otimes\lvert\mathrm{GHZ}\rangle$	N_{L}^2	$a^2N^2/2$
$\lvert\mathrm{D}_{N_{\mathrm{L}}}\rangle_x\otimes\lvert\mathrm{D}_{N_{\mathrm{L}}}\rangle_x$	$N_{\mathrm{L}}(N_{\mathrm{L}}+2)/2$	$a^2N(N+4)/4$

Note that $N_{\mathrm{L}}=N/2$.

Single ensemble for gradient estimation - Results for various spin states



Rigth: A cloud of atoms in a gradient magnetic field pointing towards the z direction.
Left: Different collective spin polarization (red) and uncertainties (green): singlet, Dicke$_z$, totally polarized, Dicke$_x$ and GHZ.



The PDF of a single ensemble of particles is permutationally invariant

$$P(\boldsymbol{x})=\tfrac{1}{N!}\sum_k\Pi_k[P(\boldsymbol{x})]\,.$$

The correlation, η, is bounded by

$$\frac{-\sigma^2}{N-1}\leqslant\eta\leqslant\sigma^2\,.$$

For states **insensitive** to the homogeneous fields, the bound can only scale with N as

$$(\Delta b_1)^{-2}|_{\max}=(\sigma^2-\eta)\sum_{n=1}^{N}\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})},j_z^{(n)}]\,.$$

For states **sensitive** to the homogeneous fields, the **shot-noise limit** can be surpassed

$$(\Delta b_1)^{-2}|_{\max}=(\sigma^2-\eta)\sum_{n=1}^{N}\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})},j_z^{(n)}]+\eta\mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})},J_z]\,.$$

Bounds for different internal states

Any **singlet state** is in the subspace of all eigenstates of J_z and J_z^2 with eigenvalue equal to zero,

$$\varrho_{\mathrm{singlet}}^{(\mathrm{s})}=\sum_{D=1}^{D_0}p_D|0,0,D\rangle\langle0,0,D|\,.$$

The precision bound is then

$$(\Delta b_1)_{\mathrm{singlet}}^{-2}|_{\max}=(\sigma^2-\eta)N\frac{4j(j+1)}{3}\,.$$

This bound can be saturated measuring the 2nd moment of J_x,

$$(\Delta b_1)^{-2}=\frac{|\partial_{b_1}\langle J_x^2\rangle|^2}{(\Delta J_x^2)^2}\,,$$

which is analytically proven to coincide with the bound for the short time limit.

For the **totally polarizedstate** along the y direction, a state sensitive to the homogeneous field, we have

$$(\Delta b_1)_{\mathrm{tp}}^{-2}|_{\max}=2\sigma^2Nj\,.$$

The **best** posible bound for a **separablestate** is computed considering that the bound is translationally invariant. Hence, we only have to compute the $\mathcal{F}_{11}$, which is maximized by the pure state

$$|\psi_{\mathrm{sep}}\rangle=\left(\tfrac{|-j\rangle+|+j\rangle}{\sqrt{2}}\right)^{\otimes N}.$$

Finally, the bound for separable states is

$$(\Delta b_1)_{\mathrm{sep}}^{-2}|_{\max}=4\sigma^2Nj^2\,.$$

The unpolarized **Dicke state**, which are symmetric eigenstates of the J_l operator are written as

$$|\mathrm{D}_N\rangle_l=\binom{N}{N/2}^{-1/2}\sum_k\mathcal{P}_k(|0\rangle_l^{\otimes N/2}\otimes|1\rangle_l^{\otimes N/2})\,.$$

We are interested in $l = x$, which is sensitive to the homogeneous fields, and $l = z$, which is not.

For both Dicke states the precision bounds are

$$(\Delta b_1)_{\mathrm{D}}^{-2}|_{\max}=(\sigma^2-\eta)N\,,$$

$$(\Delta b_1)_{\mathrm{D},x}^{-2}|_{\max}=(\sigma^2-\eta)N+\eta\frac{N(N+2)}{2}\,.$$

To conclude, we present the **GHZ state**

$$|\mathrm{GHZ}\rangle=\frac{|00\cdots00\rangle+|11\cdots11\rangle}{\sqrt{2}}\,.$$

For estimating the homogeneous field this state reaches the Heisenberg limit, $(\Delta b_1)^{-2}$ ~ N^2, and hence, the bound for the gradient estimation is

$$(\Delta b_1)_{\mathrm{GHZ}}^{-2}|_{\max}=(\sigma^2-\eta)N+\eta N^2\,.$$

which scales with N^2 whenever η is constant.

All the bound in this section are saturable, as it was shown in "Measurements compatibility".

Bose-Einstein condensates

In a BEC all the particles share the same spatial state, $\varrho_{\mathrm{BEC}}^{(\mathrm{x})}=(|\Psi\rangle\langle\Psi|)^{\otimes N}$.

Since the bound is invariant under th translations of the system, we only have to compute $\mathcal{F}_{11}$ even if the state is sensitive to the homogeneous fields,

$$\mathcal{F}_{11}=4\sigma^2\,\mathrm{tr}\left[\sum_n(j_z^{(n)})^2\varrho^{(\mathrm{s})}\right]\,.$$

The state that maximizes the QFI is the totally polarized state along the z direction, $|\Psi\rangle_{\mathrm{opt,BEC}}=|j\rangle^{\otimes N}$, from which we obtain the bound

$$(\Delta b_1)^{-2}|_{\max}=4\sigma^2Nj^2\,.$$

Note that the totally polarized state is insensitive to the homogeneous fields, so the bound can be saturated and no state can surpass it.

Conclusions

- We obtained **general formulas** to compute the precision bounds for gradient magnetometry for spin-chains, double-wells, atomic single clouds and BECs.
- These bounds are based on the internal state of the system.
- Among the bounds we presented for an atomic clound, there is the bound for the best separable state.
- We proved that all bounds in this work are saturable, in particular if the internal spin state is permutationally invariant.

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