How energy conservation limits our measurements

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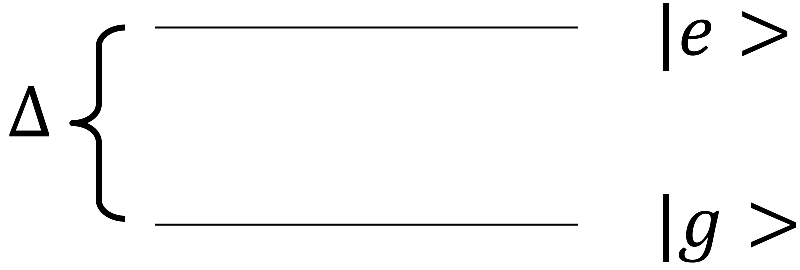
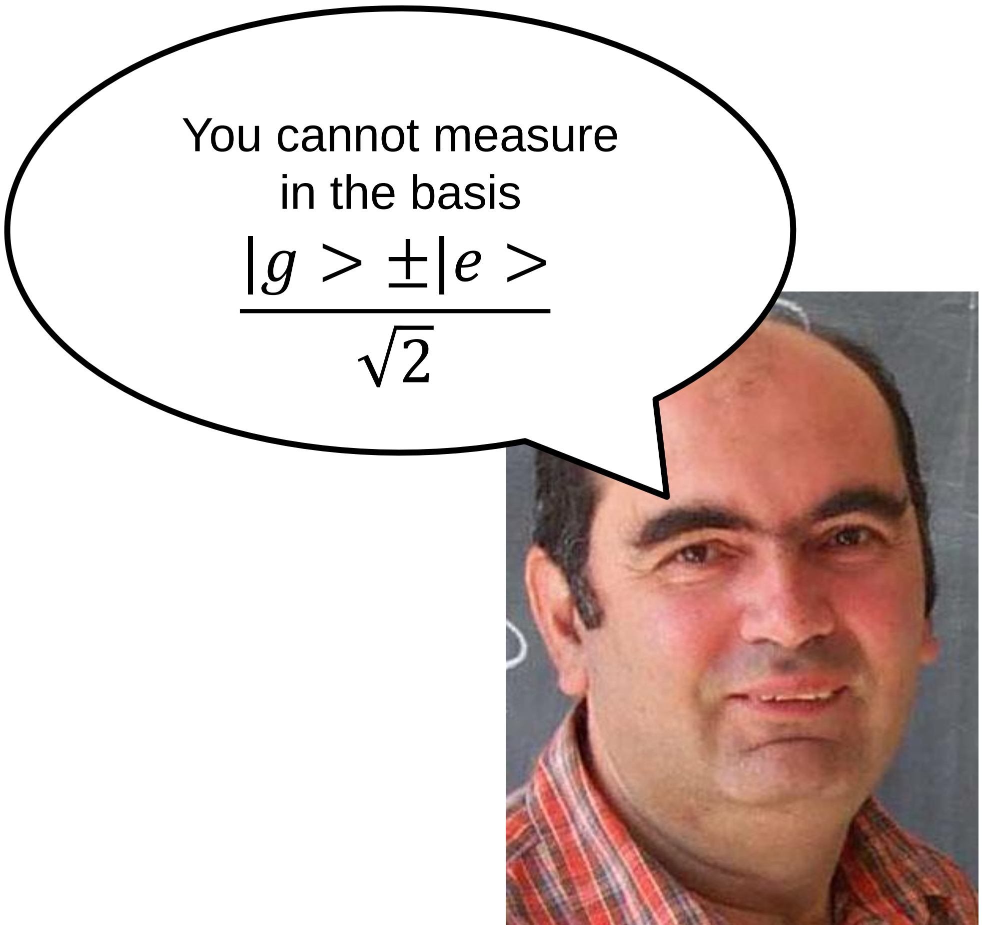
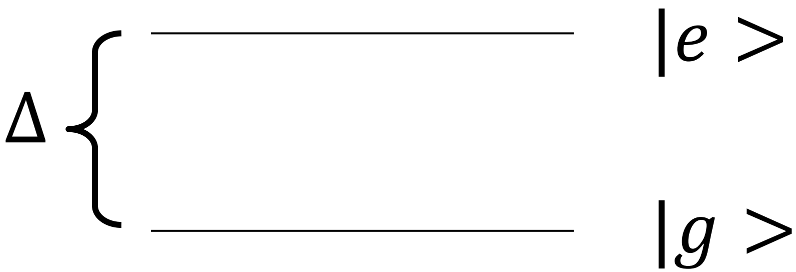
arXiv:1211.2101

I can explain everything

Everything is Sandu’s fault!


Δ
$|e>$
$|g>$

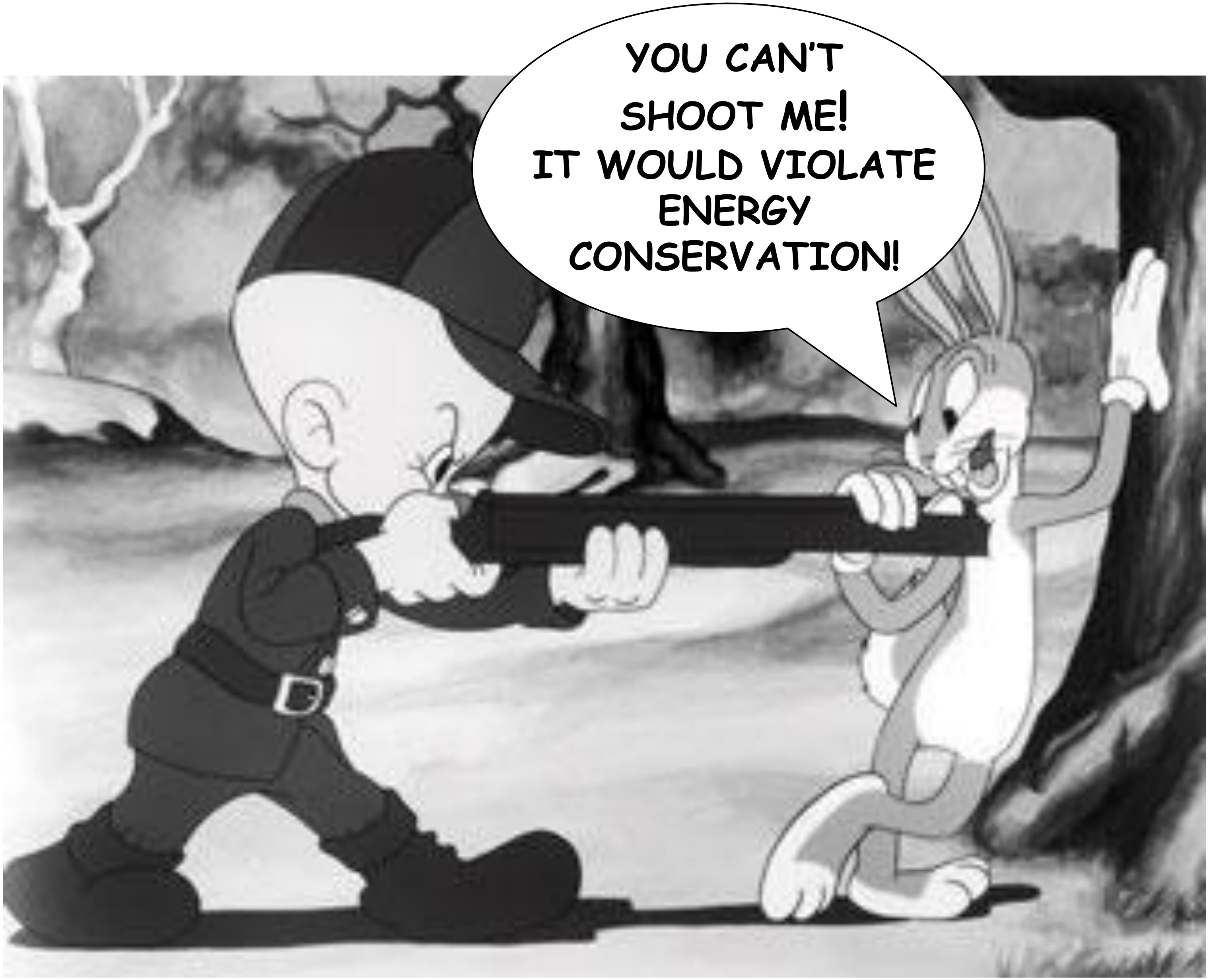

You cannot measure
in the basis
$\frac{|g> \pm |e>}{\sqrt{2}}$

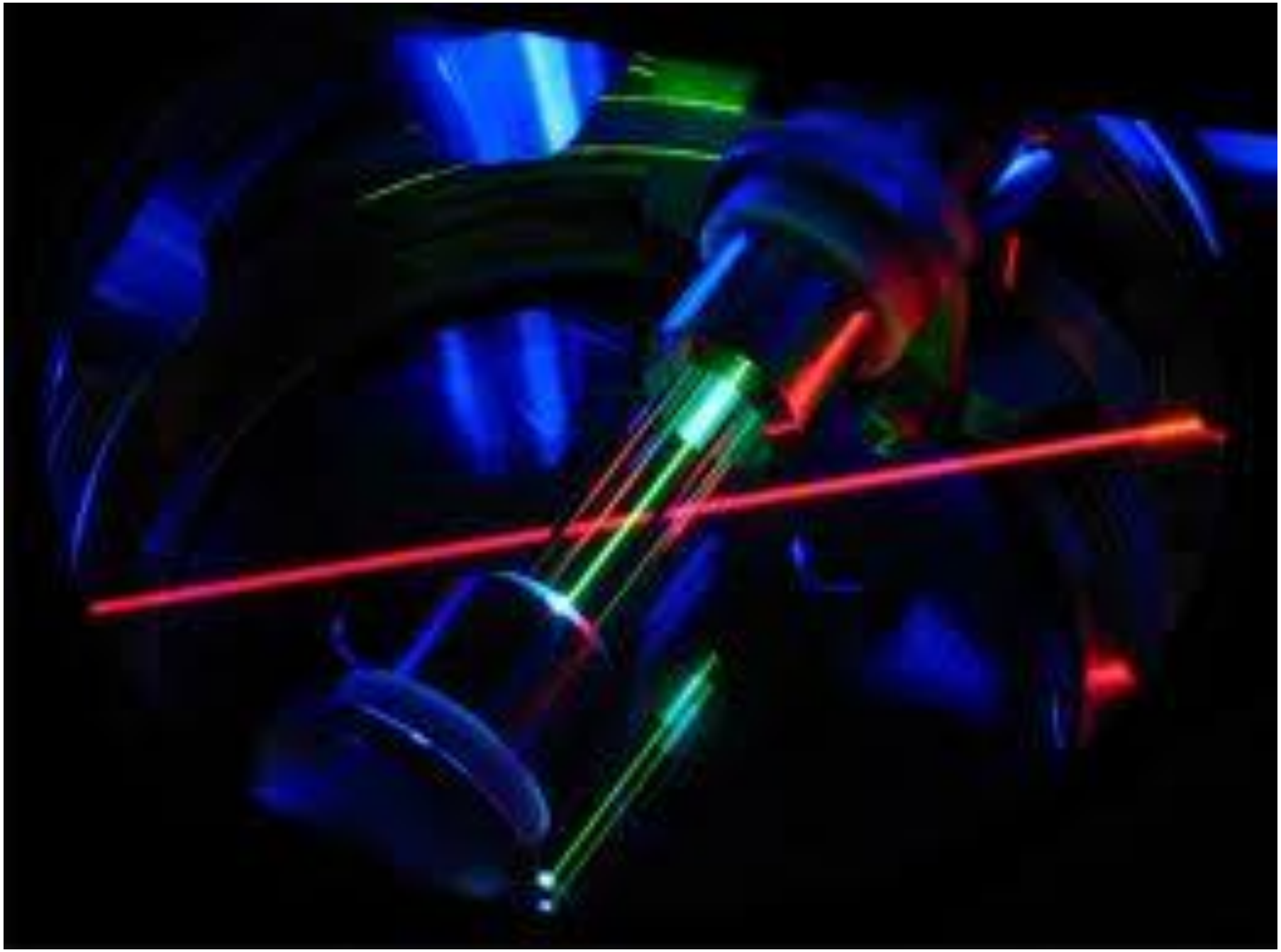


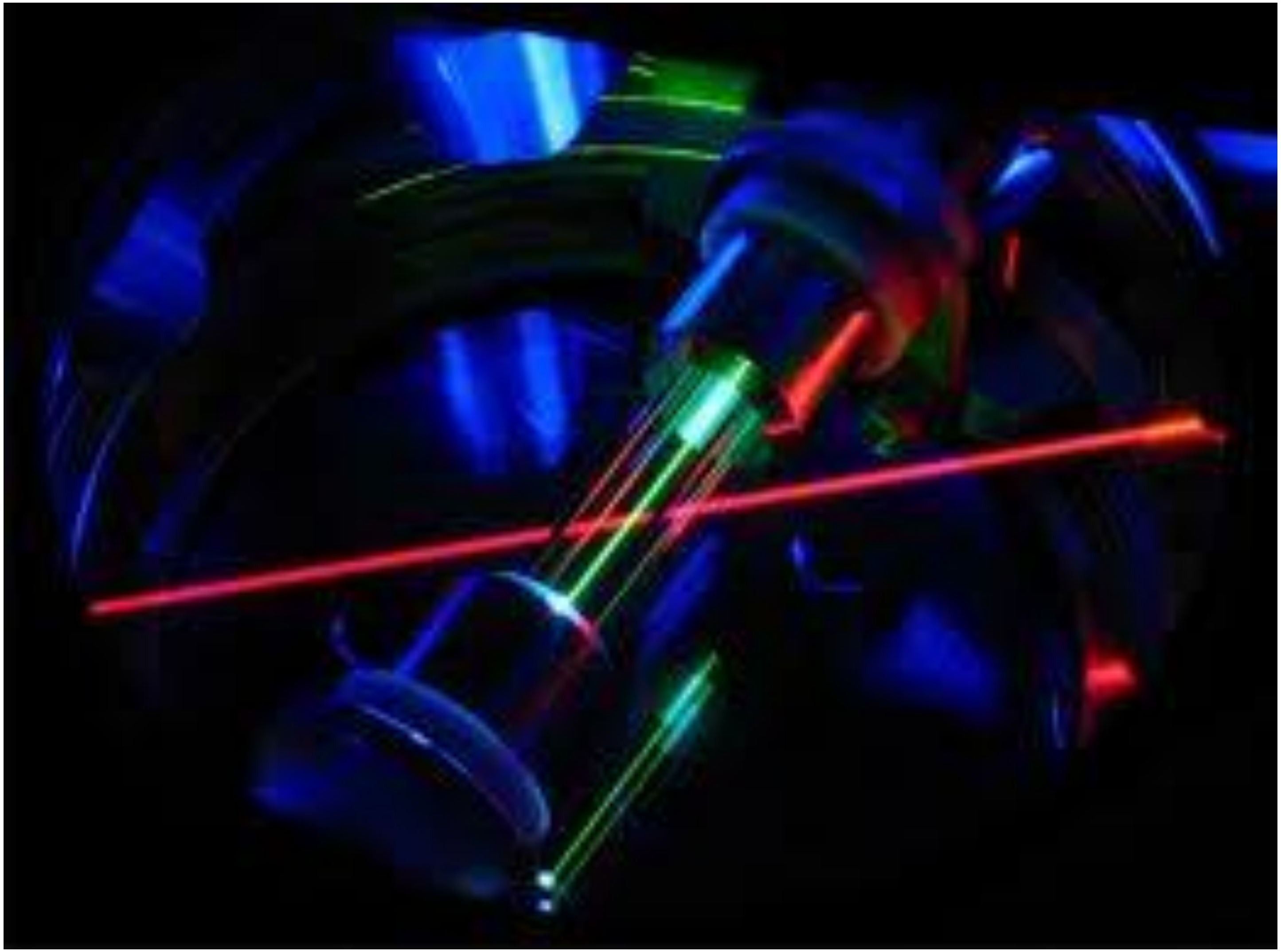
$$|g> \longrightarrow \frac{|g> \pm |e>}{\sqrt{2}}$$

$$E = 0 \qquad\qquad E = \Delta/2$$

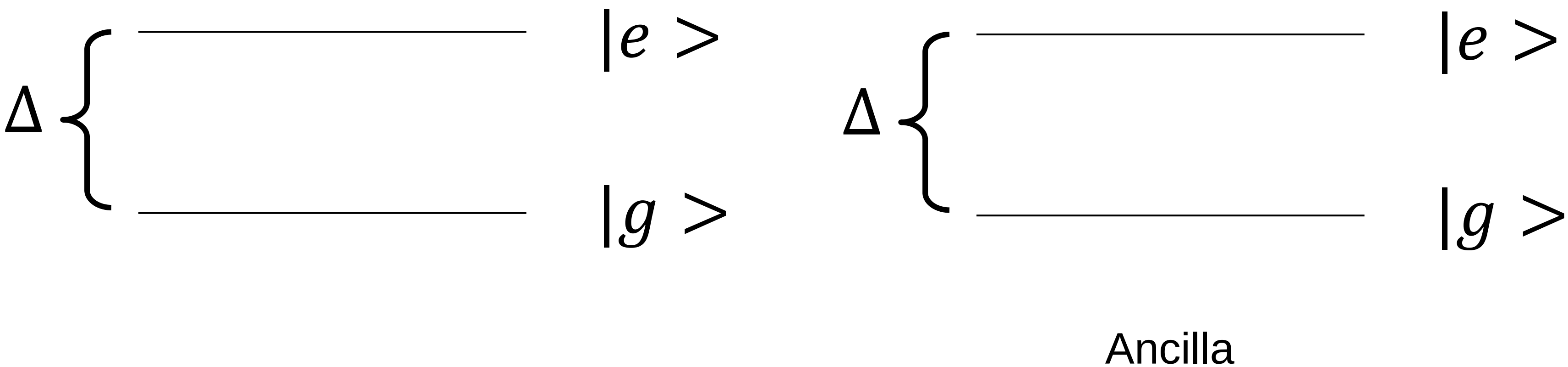
Violates energy conservation!!!


YOU CAN'T SHOOT ME! IT WOULD VIOLATE ENERGY CONSERVATION!





Lots of approximations, infinite energy…



Δ
|e >
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Δ
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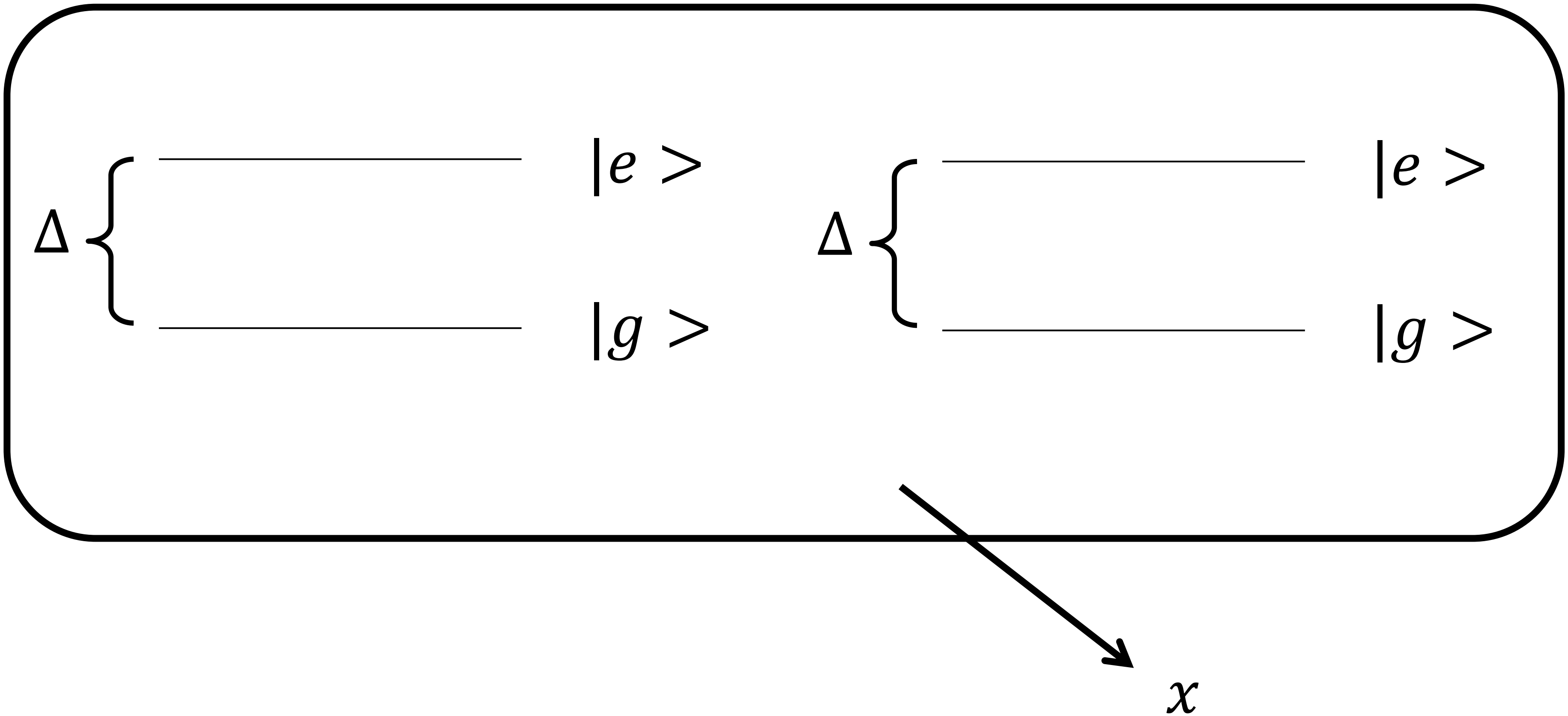
Ancilla


Δ
|e >
|g >
Δ
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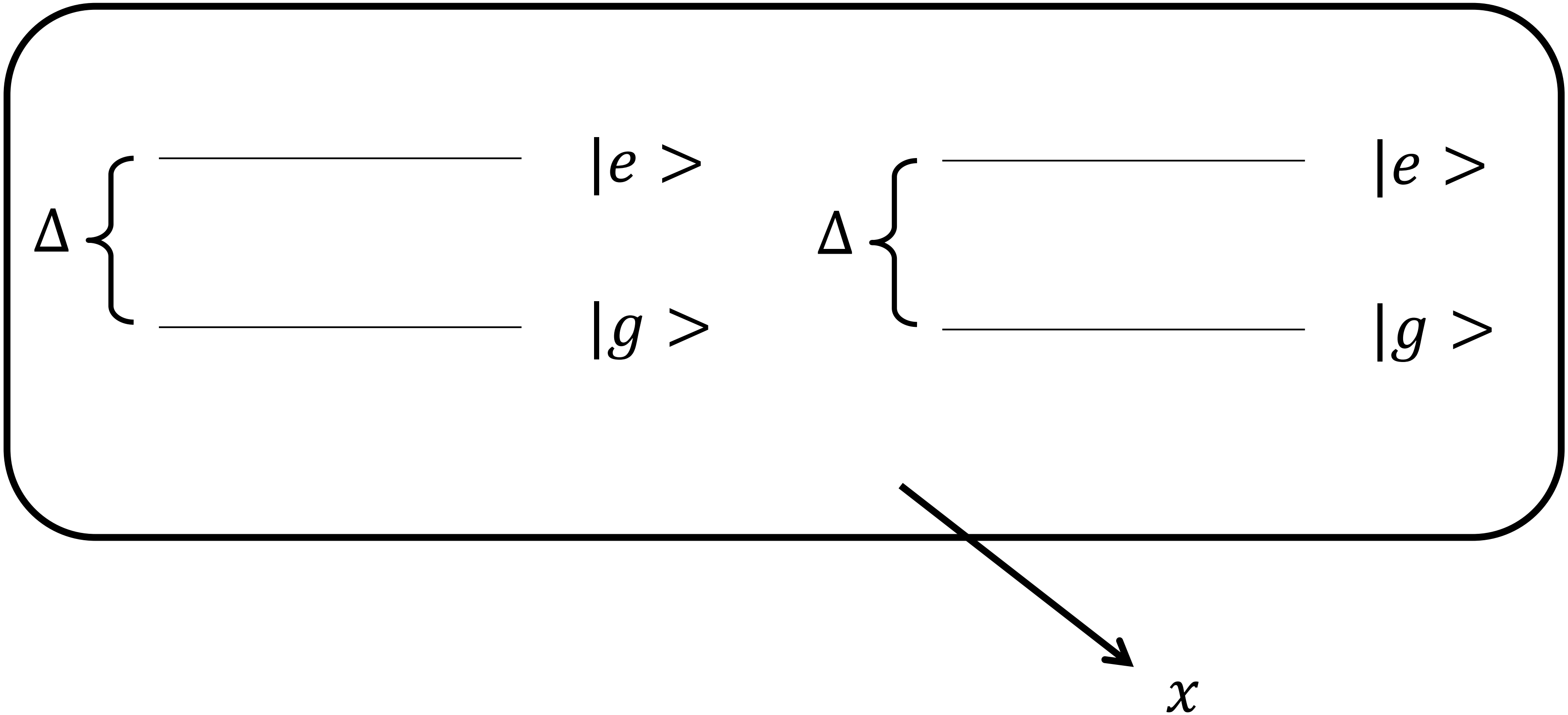
x

Energy conservation demands that

$$H_T = H_s \otimes \mathbb{I} + \mathbb{I} \otimes H_s$$



Energy conservation demands that

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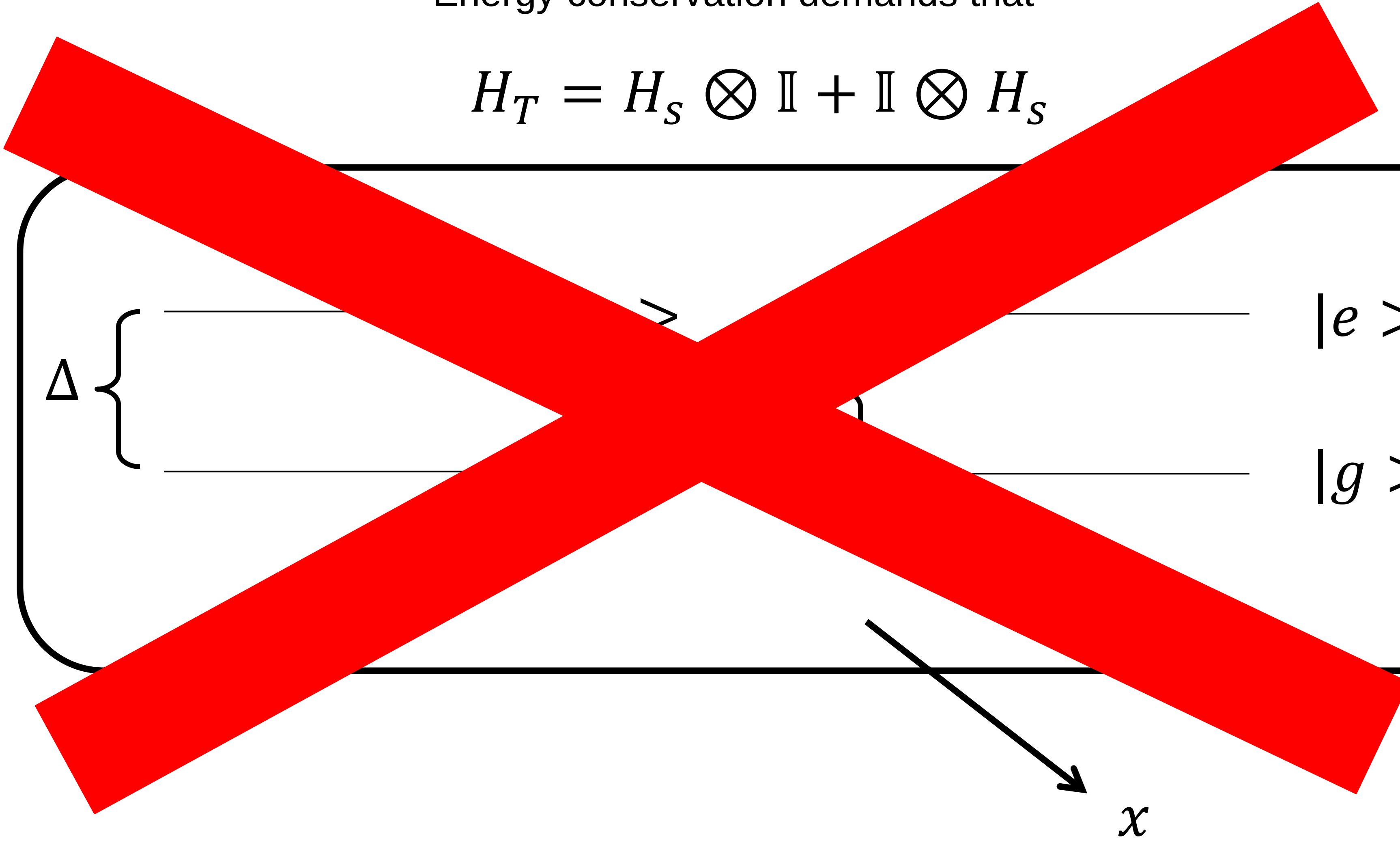

Previous work

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H. Araki and M. M. Yanase, Phys. Rev. 120, 622626 (1960).

M. M. Yanase, Phys. Rev. 123, 666 (1961).

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M. Ozawa, Phys. Rev. Lett. 88, 050402 (2002).

T. Karasawa, J. Gea-Banacloche and M. Ozawa J. Phys. A: Math. Theor. 42, 225303 (2009).

J. Gea-Banacloche and M. Ozawa, J. Opt. B: quantum Semiclass. Opt. 7, S326 (2005).

How conservation laws limit unitary evolution, general uncertainty relation

S. D. Bartlett, T. Rudolph, R. W. Spekkens and P. S. Turner, New J. Phys. 11, 063013 (2009).

G. Gour, I. Marvian and R. W. Spekkens, Phys. Rev. A 80, 012307 (2009).

Resource theories

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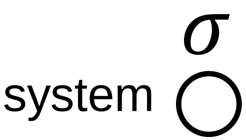
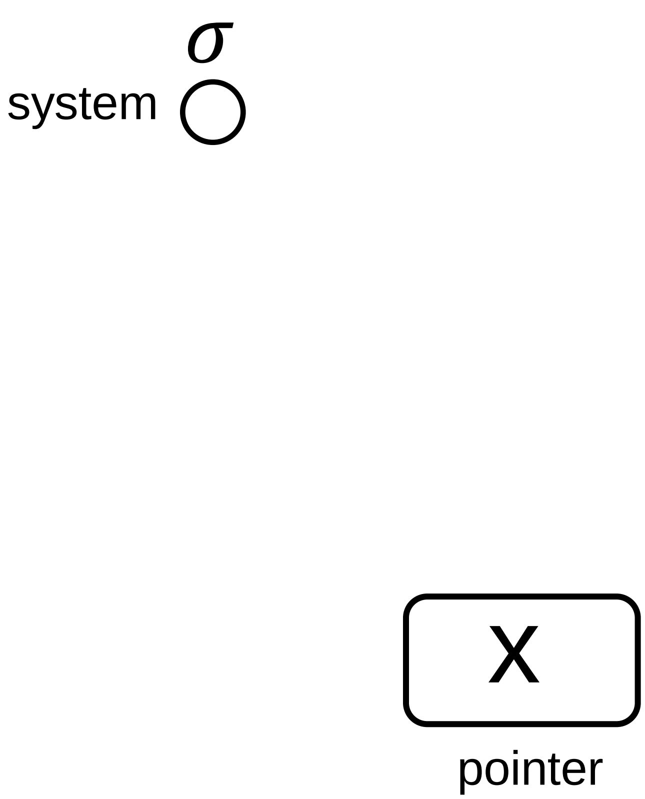
Studies state estimation problems under conservation laws

The measurement model

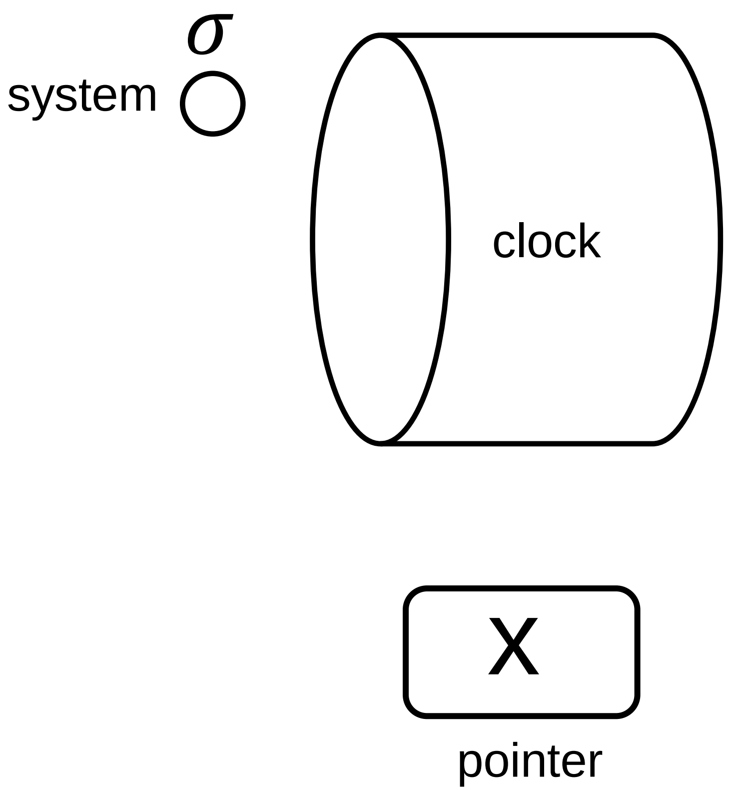
Classical measurement model



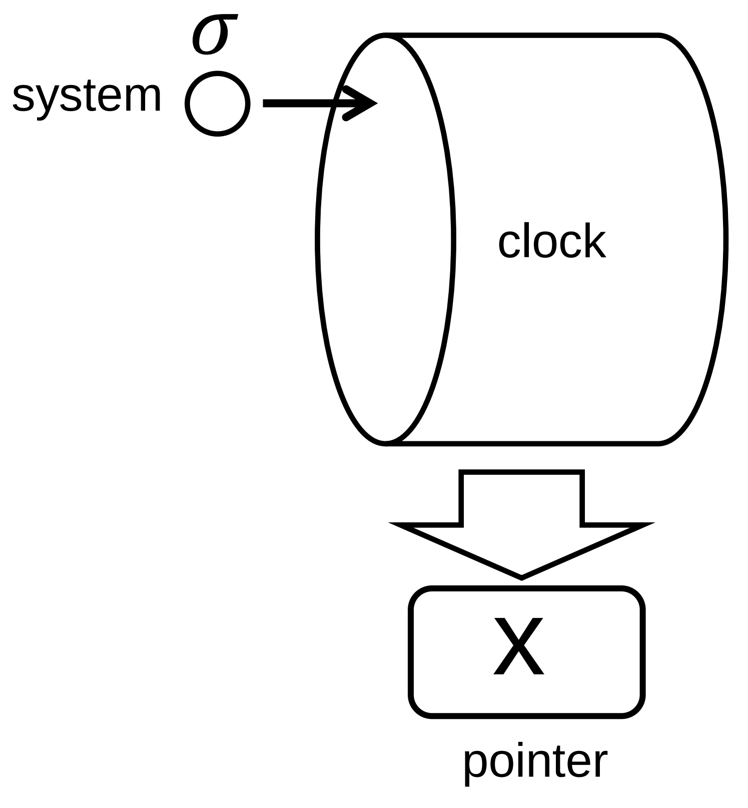
Classical measurement model



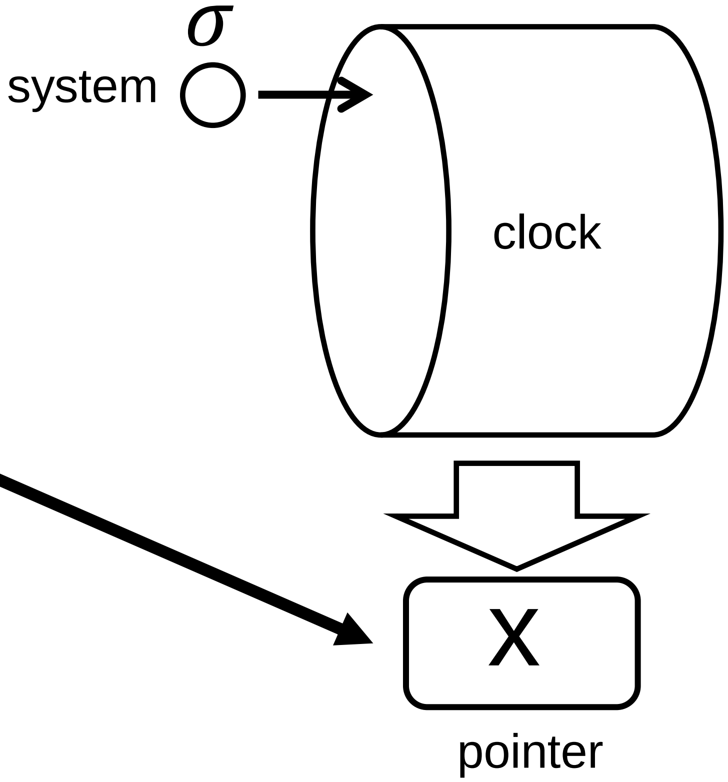
Classical measurement model



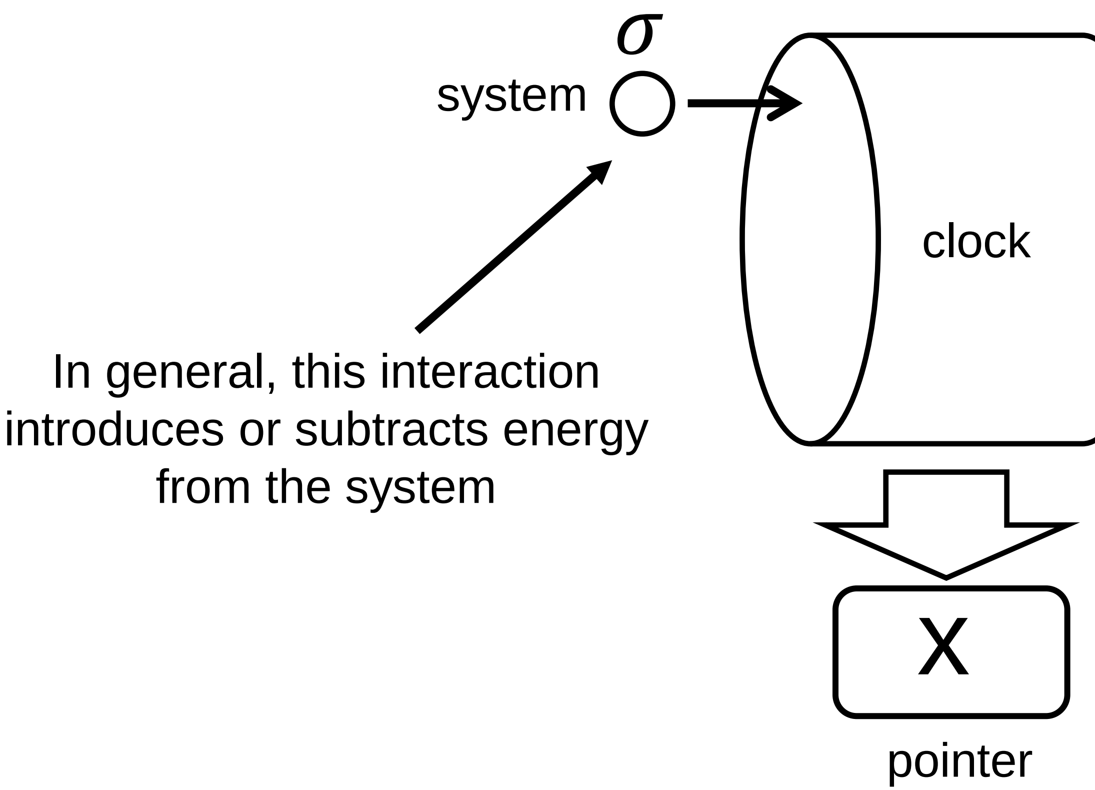
Classical measurement model



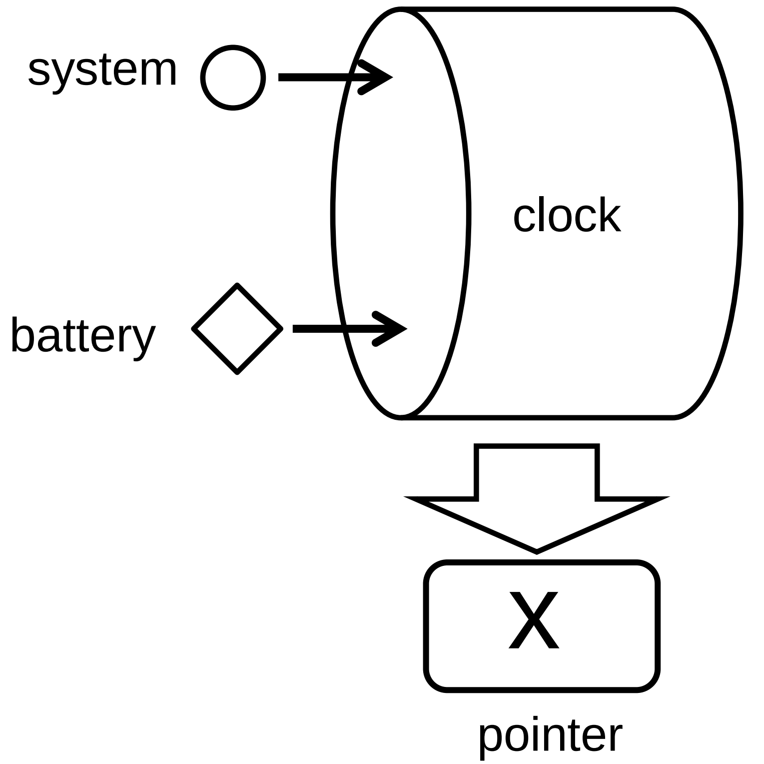
Classical measurement model



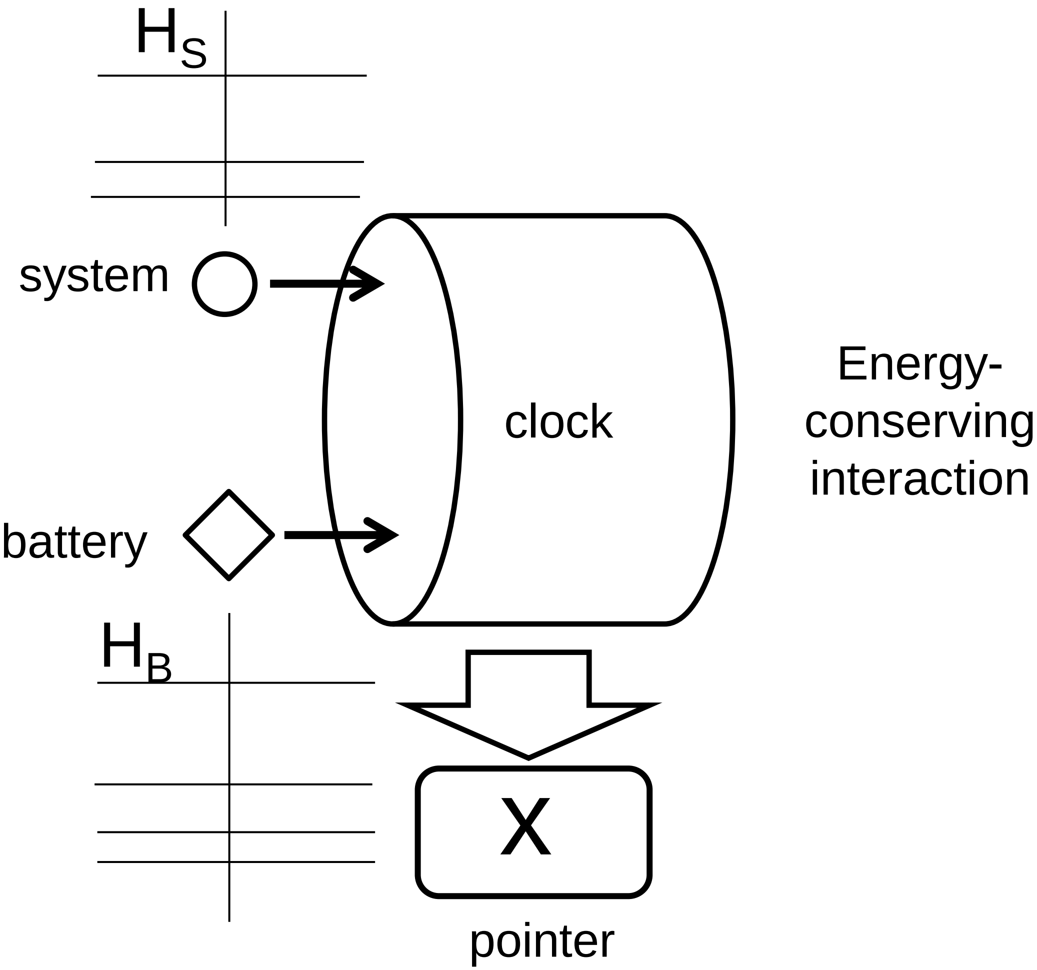
Classical measurement model



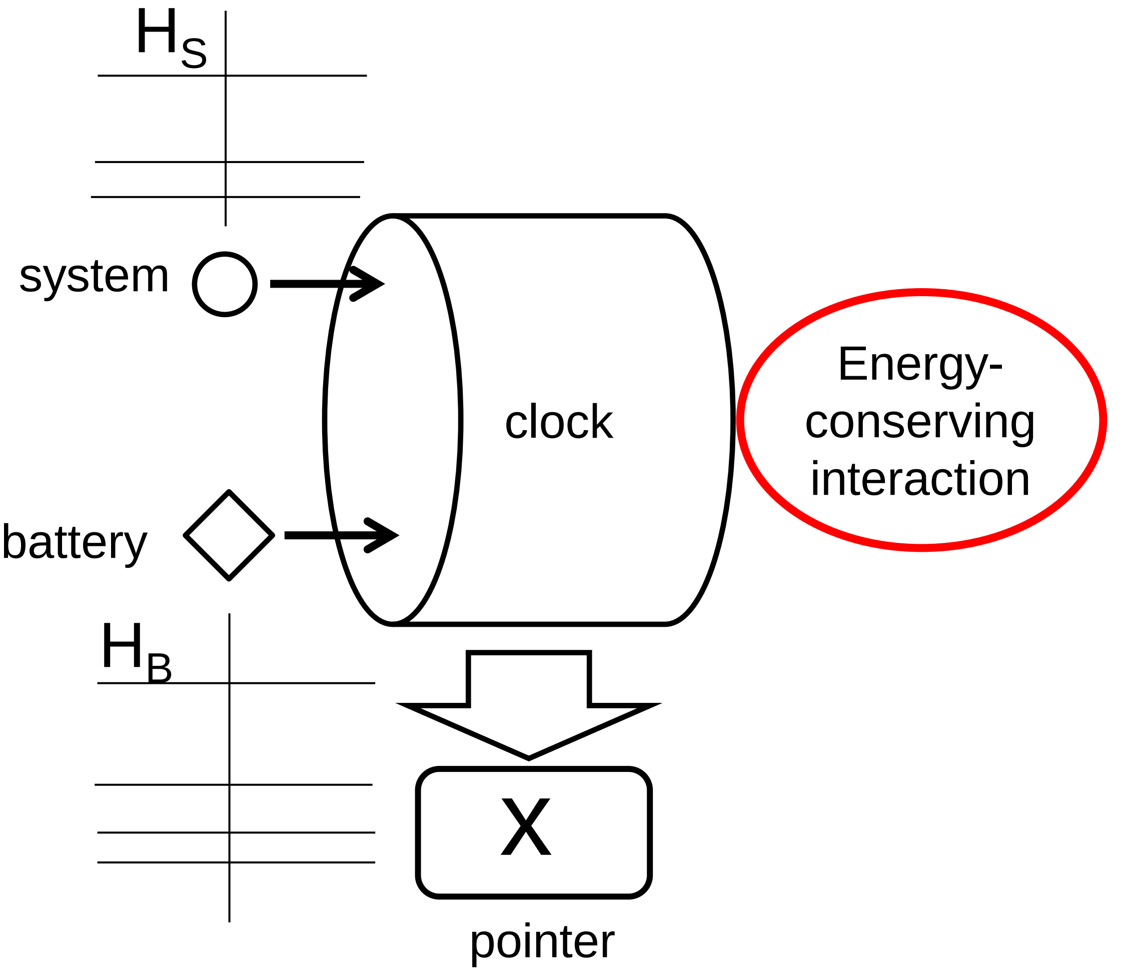
Energy-conserving measurement model

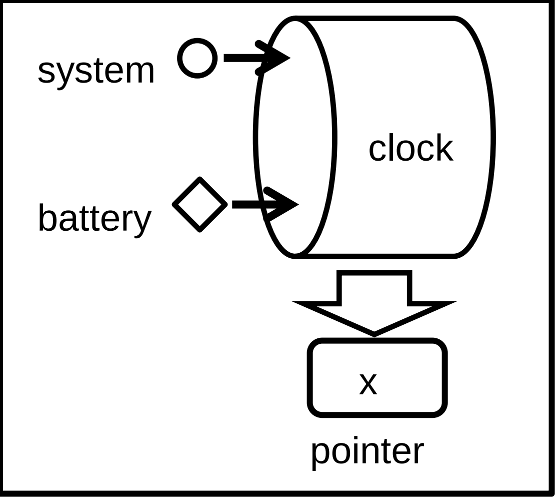


Energy-conserving measurement model



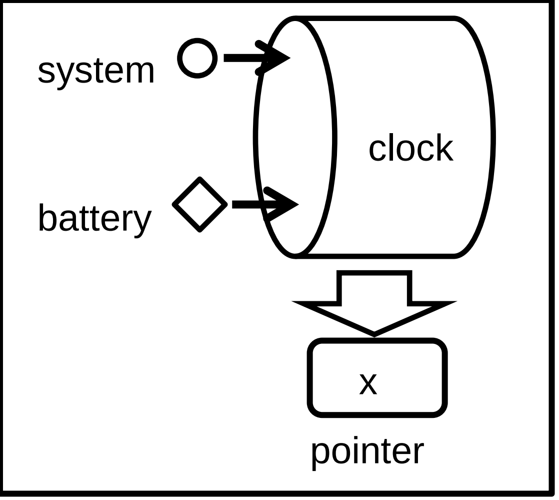
Energy-conserving measurement model



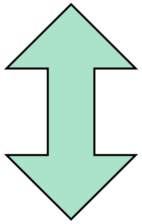


$$< \phi_{SBP} \, |U^* H_T U| \phi_{SBP} > = < \phi_{SBP} \, |H_T| \phi_{SBP} >$$

$$H_T = H_s \otimes \mathbb{I}_{BP} + \mathbb{I}_S \otimes H_B \otimes \mathbb{I}_P$$

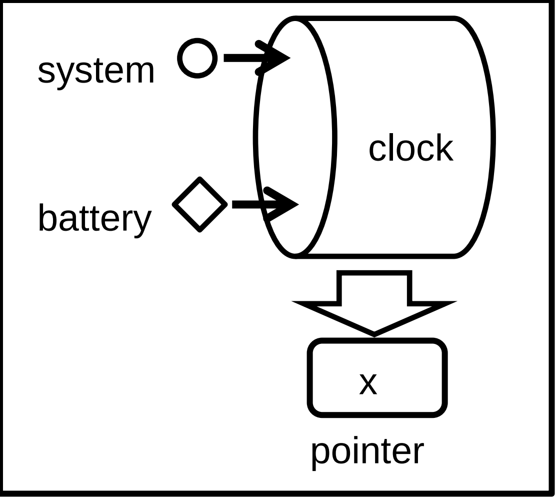


$$< \phi_{SBP} |U^* H_T U| \phi_{SBP} > = < \phi_{SBP} |H_T| \phi_{SBP} >$$



$$[U, H_T] = 0$$

$$H_T = H_s \otimes \mathbb{I}_{BP} + \mathbb{I}_S \otimes H_B \otimes \mathbb{I}_P$$



$$< \phi_{SBP} |U^* H_T U| \phi_{SBP} > = < \phi_{SBP} |H_T| \phi_{SBP} >$$

$$\Updownarrow$$

$$[U, H_T] = 0$$

We do not want the pointer to play the role of the battery!!

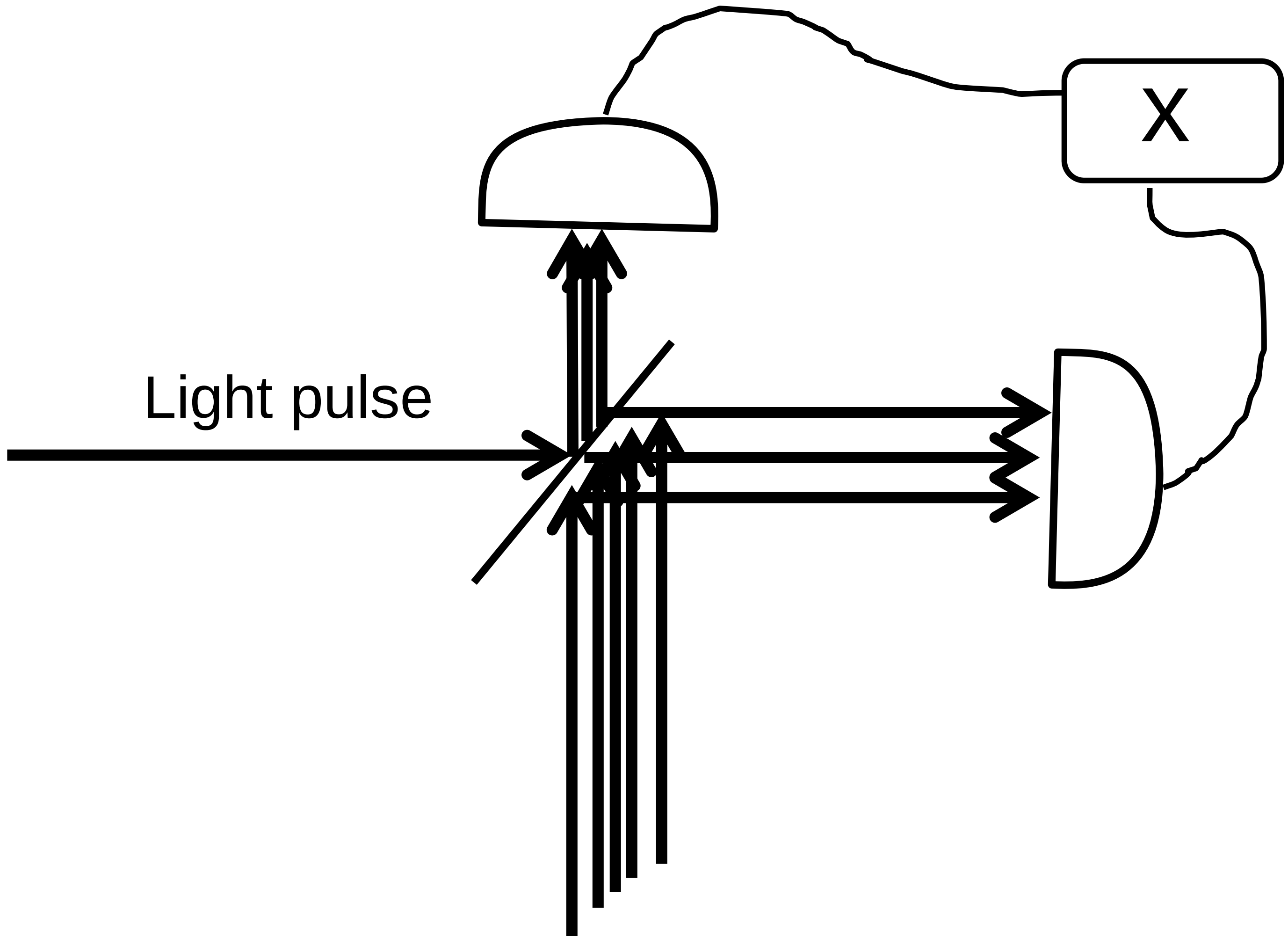
$$H_T = H_s \otimes \mathbb{I}_{BP} + \mathbb{I}_S \otimes H_B \otimes \mathbb{I}_P \Rightarrow H_P = 0$$

Example: homodyne measurements in quantum optics

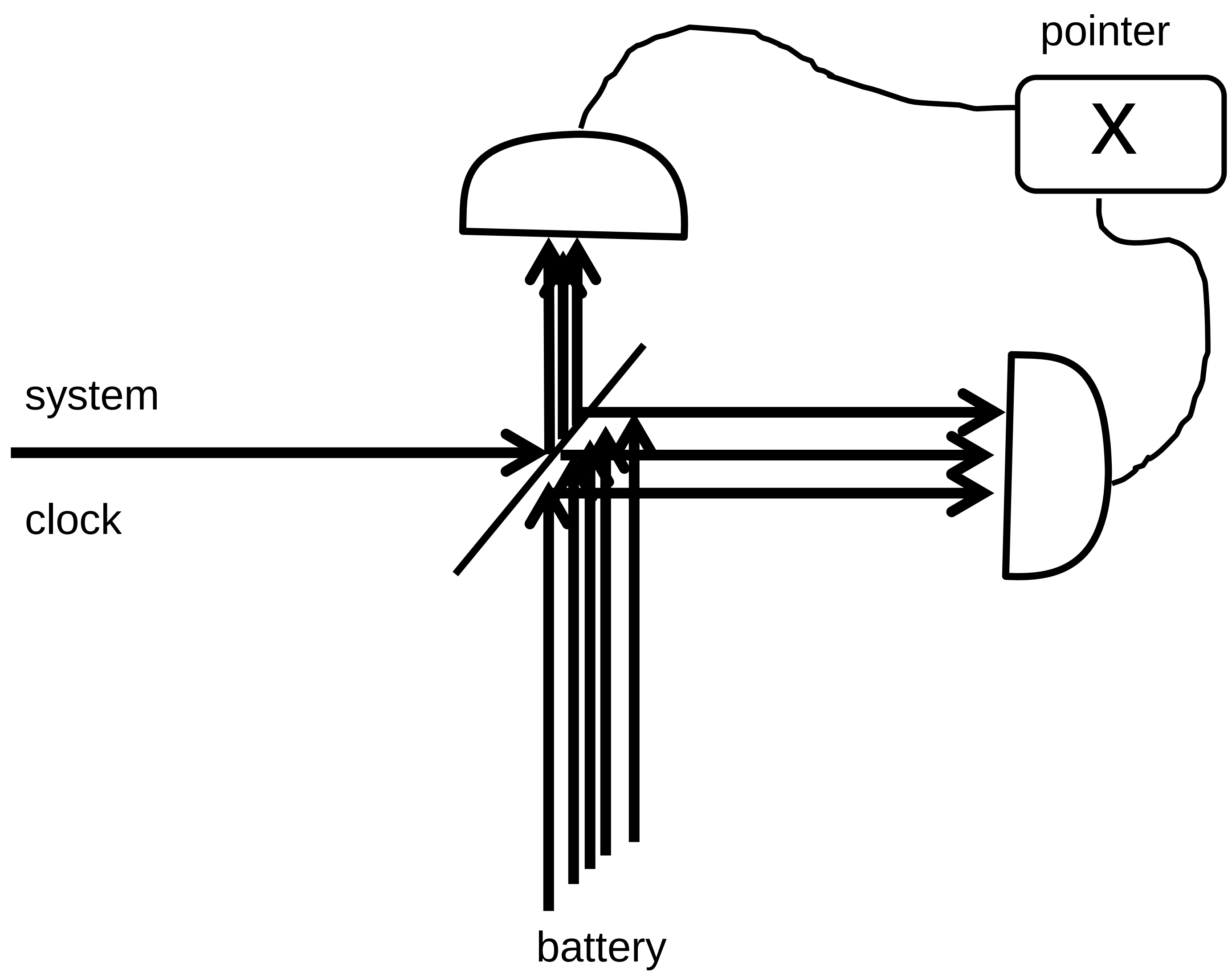
Light pulse

Aim: measure $\frac{a+a^{t}}{\sqrt{2}}$

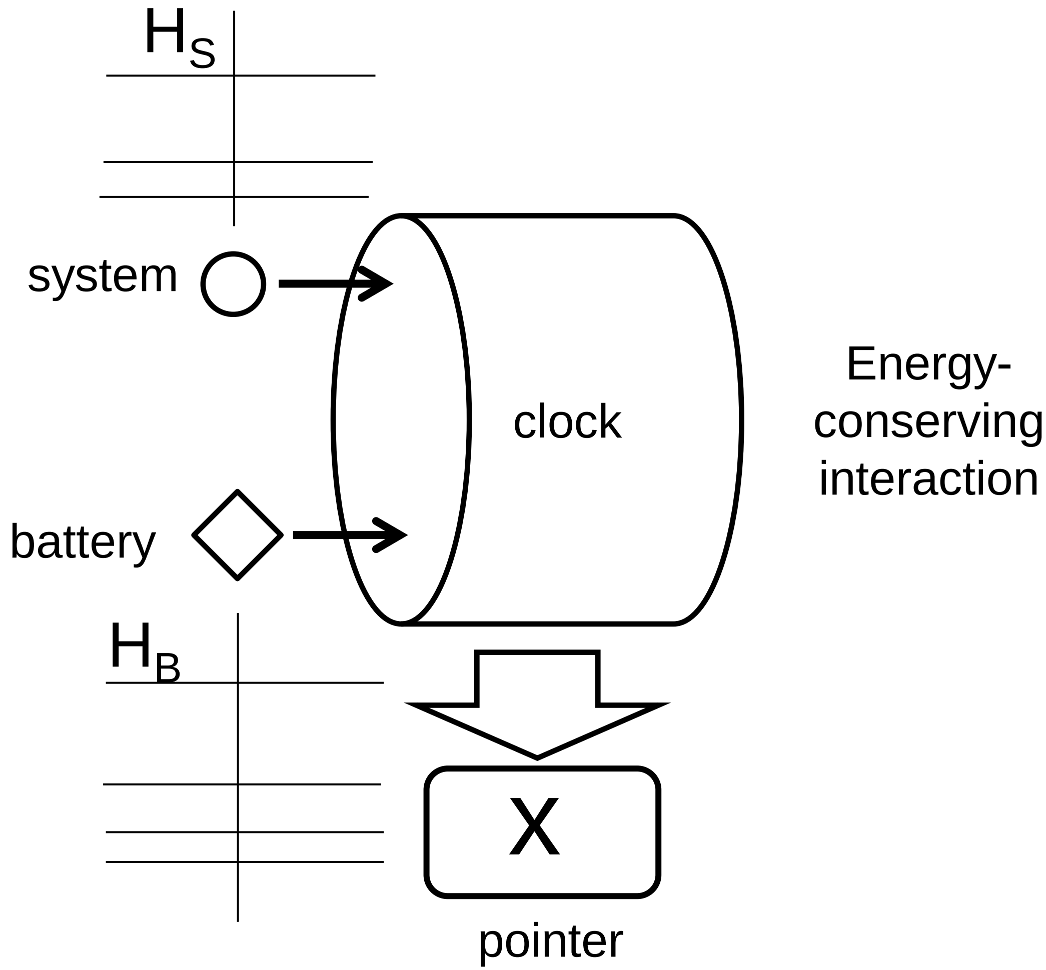
Example: homodyne measurements in quantum optics



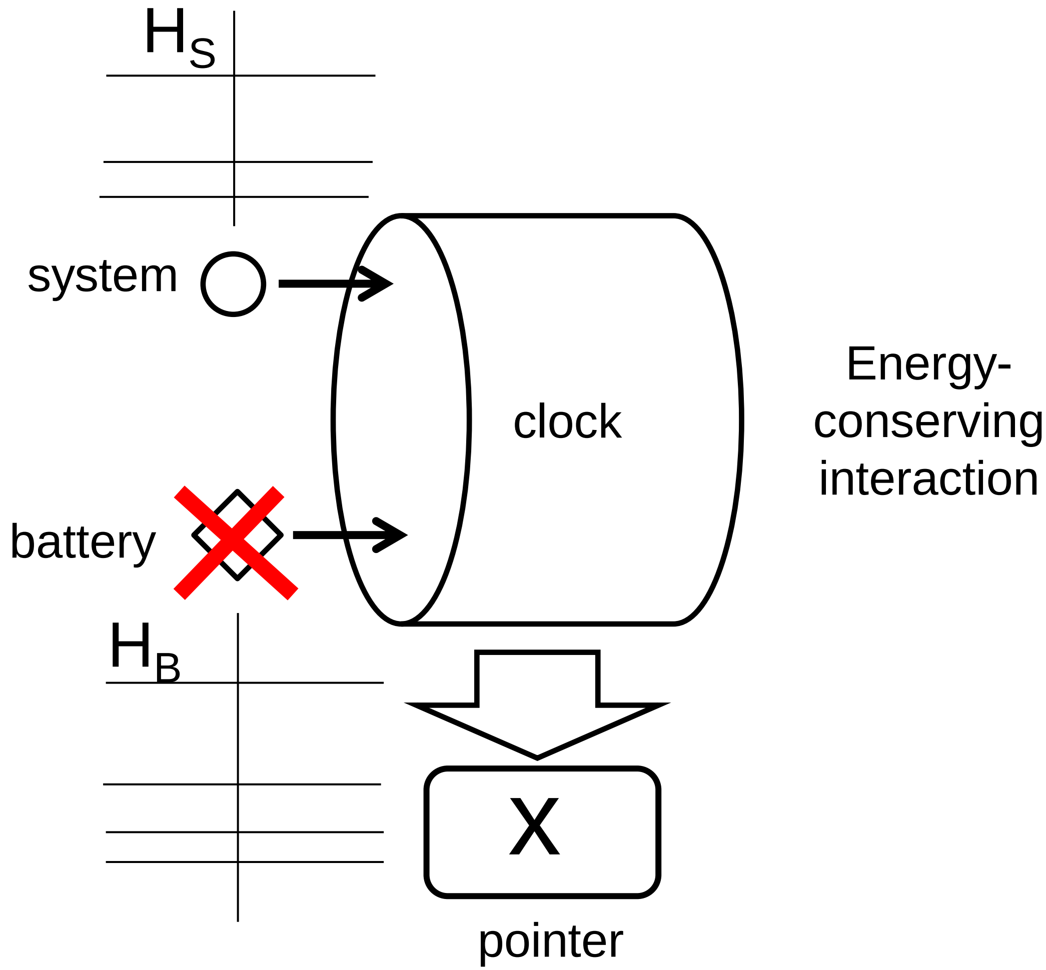
Example: homodyne measurements in quantum optics

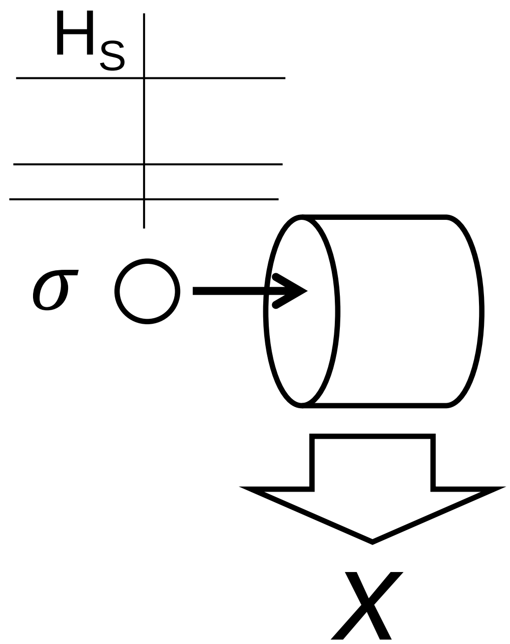


Energy-conserving measurement model

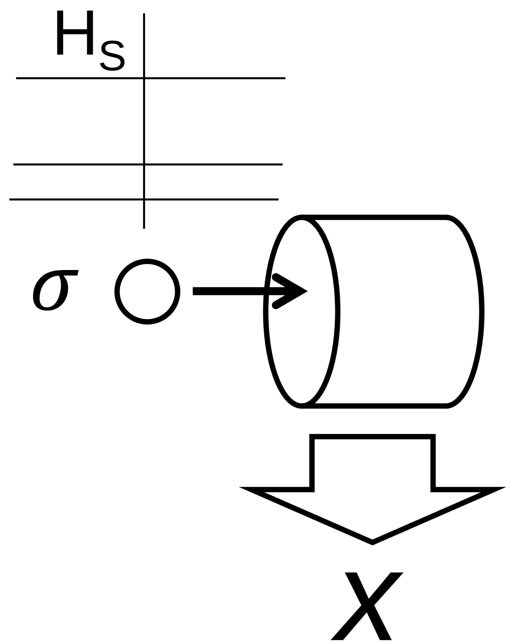


Energy-conserving measurement model



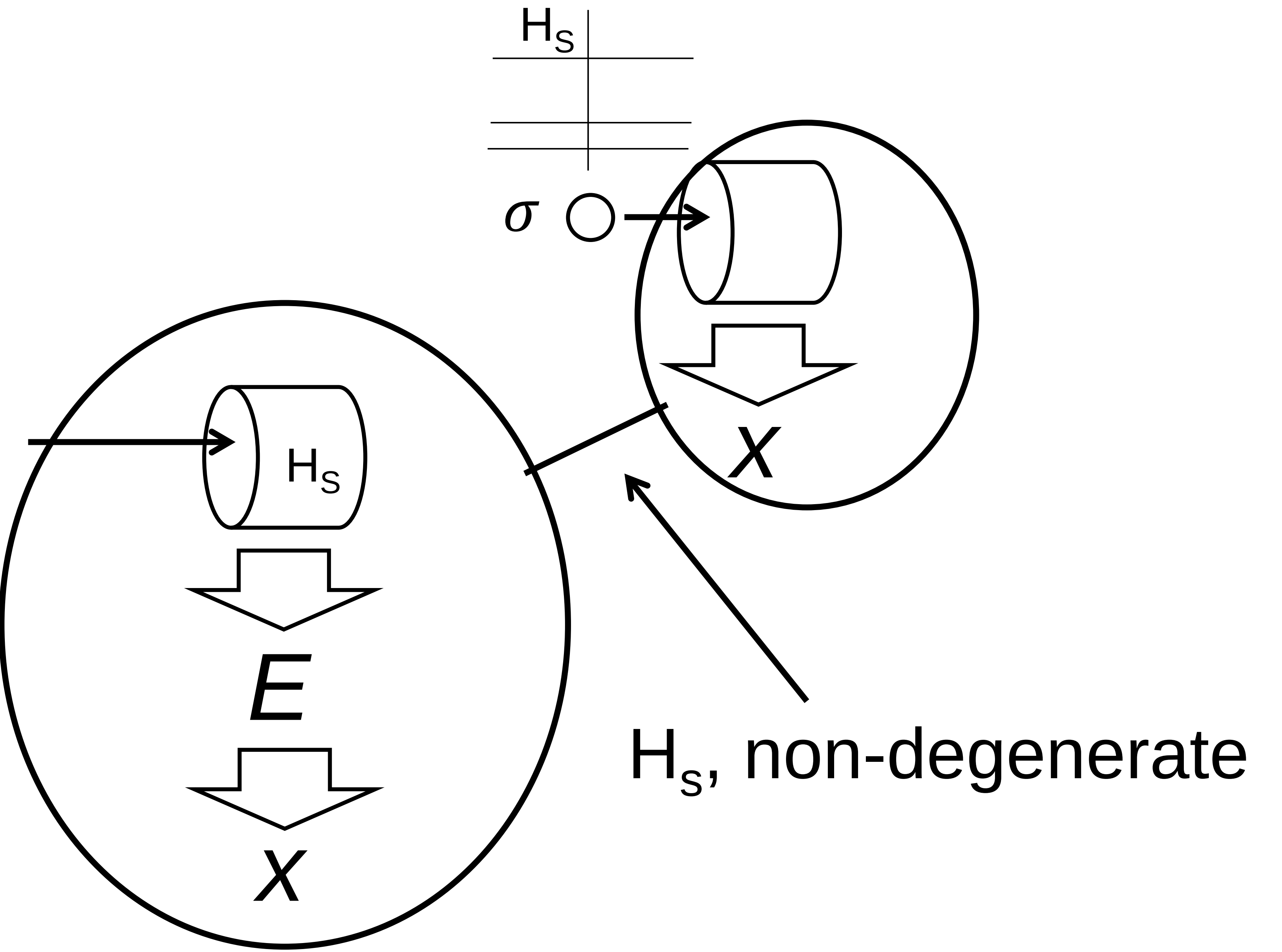


$$p(x) = tr(\sigma M_x), M_x \geq 0, \sum_x M_x = \mathbb{I}$$

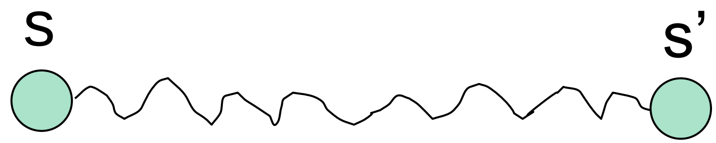


$$p(x) = tr(\sigma M_x), M_x \geq 0, \sum_x M_x = \mathbb{I}$$

$$[M_x, H_s] = 0$$

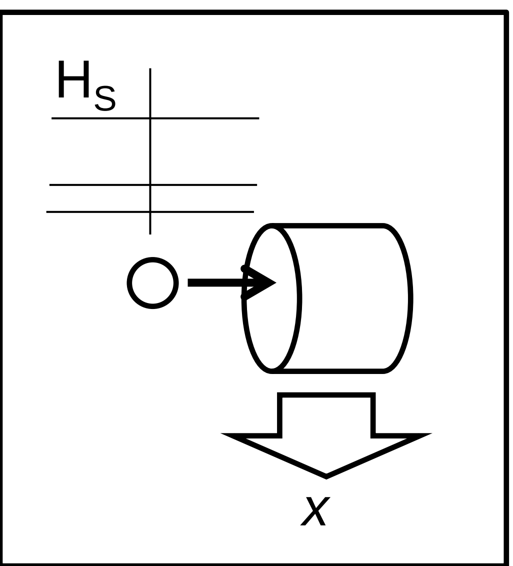


H_S
σ
X
H_S
E
X
H_s, non-degenerate

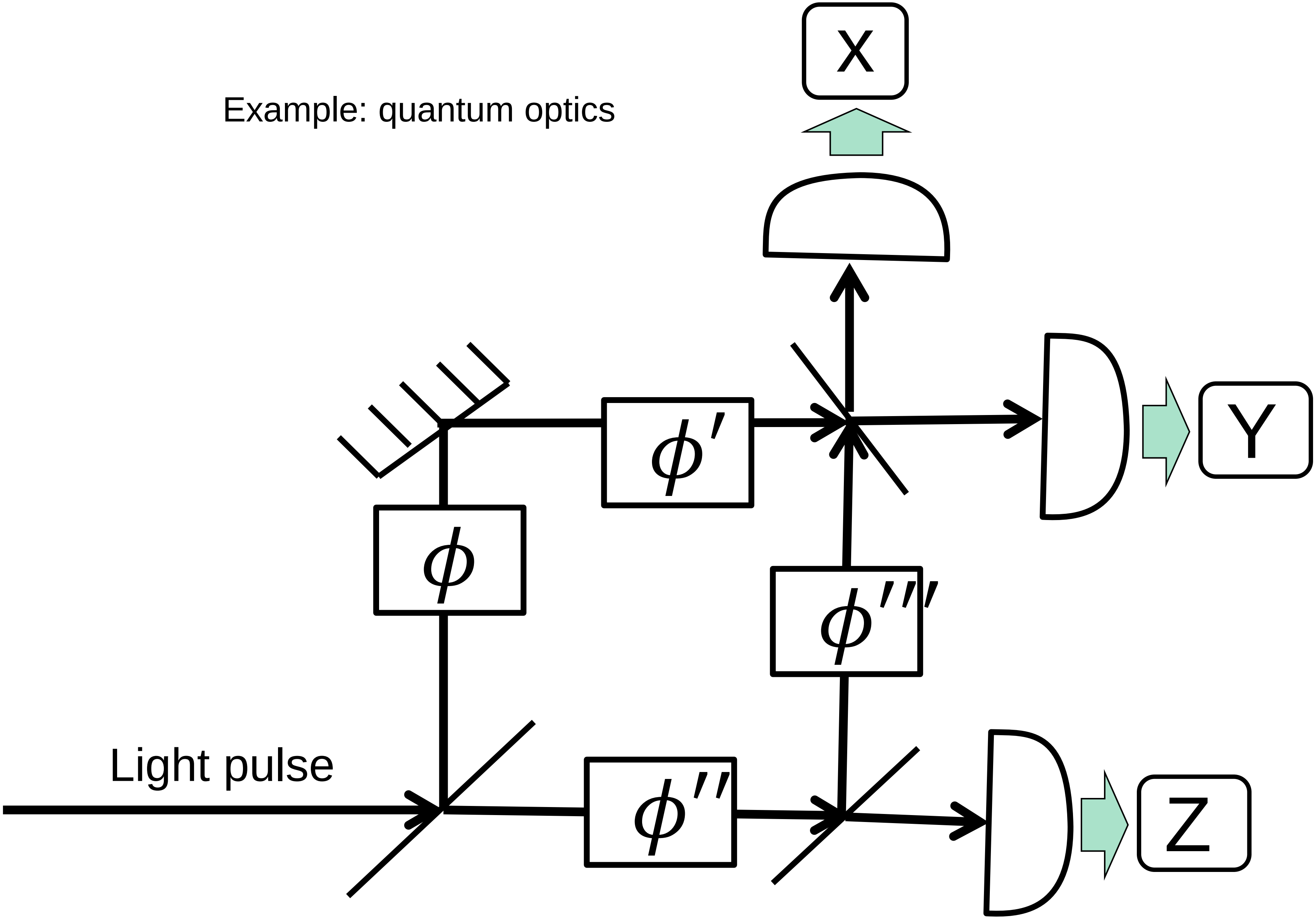


s, s’ cannot

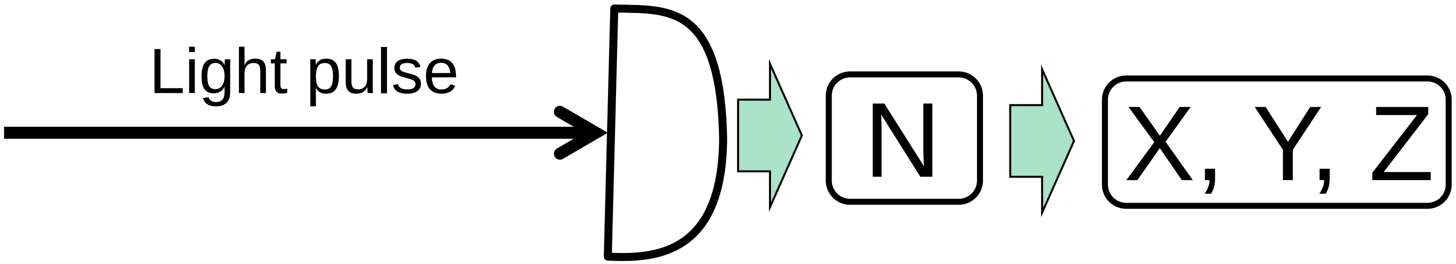
- violate Bell inequalities
- prove that their state is entangled



Example: quantum optics

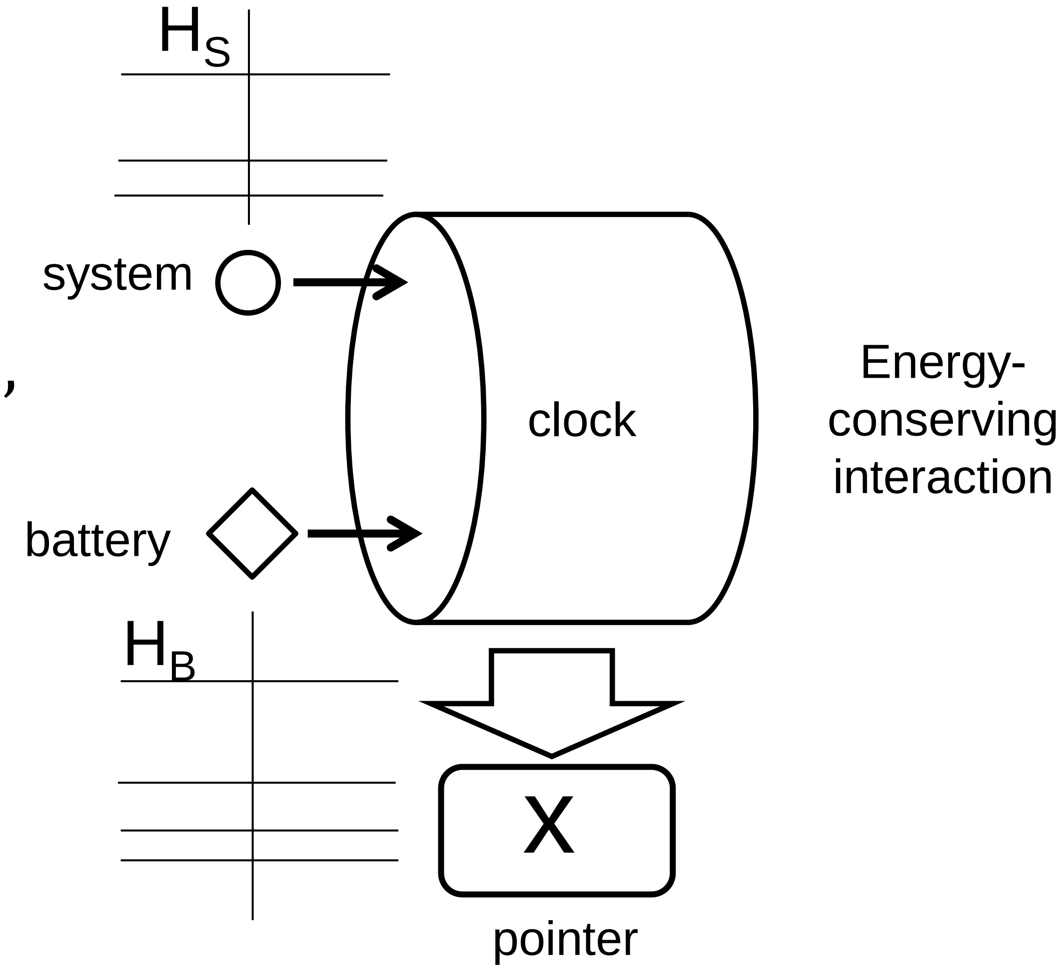


Example: quantum optics

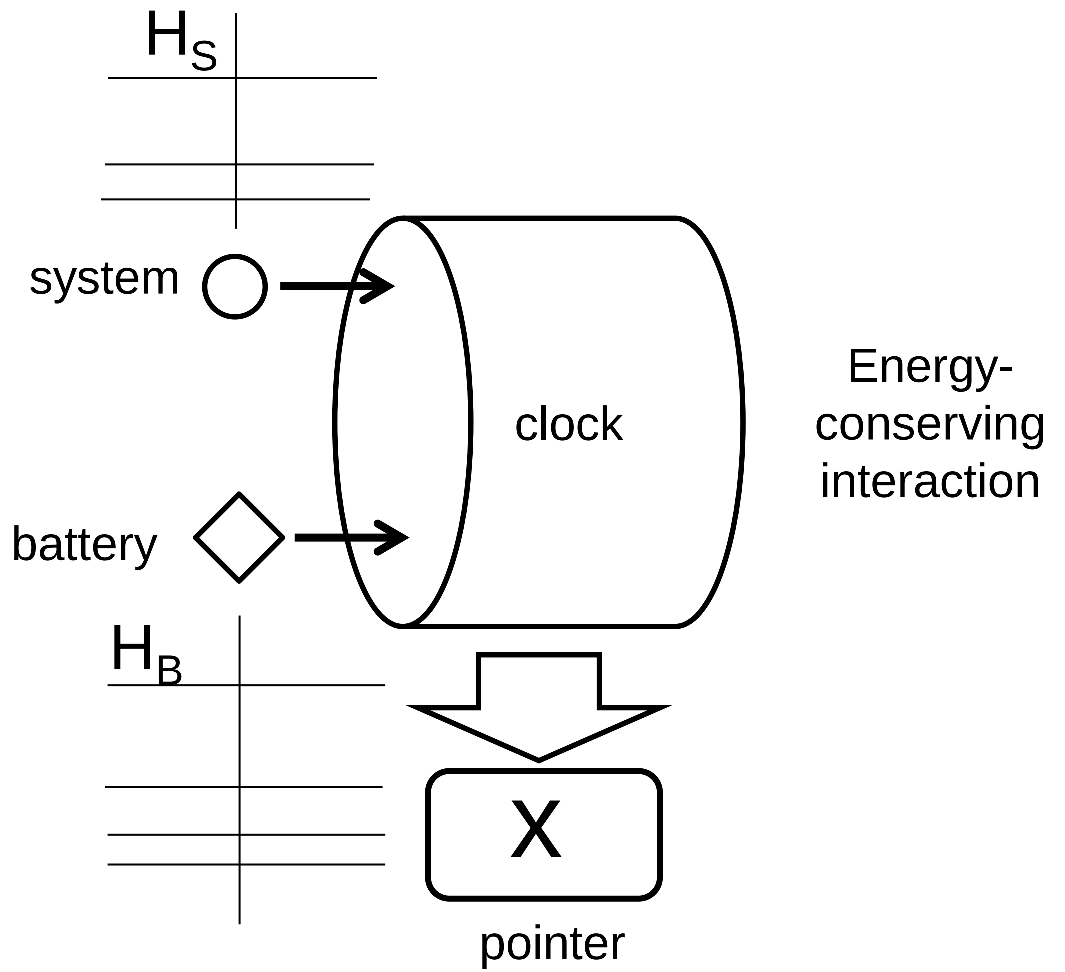


$$p(x) = tr((\sigma \otimes \varrho_B) M_x),$$

$$[M_x, H_T] = 0$$

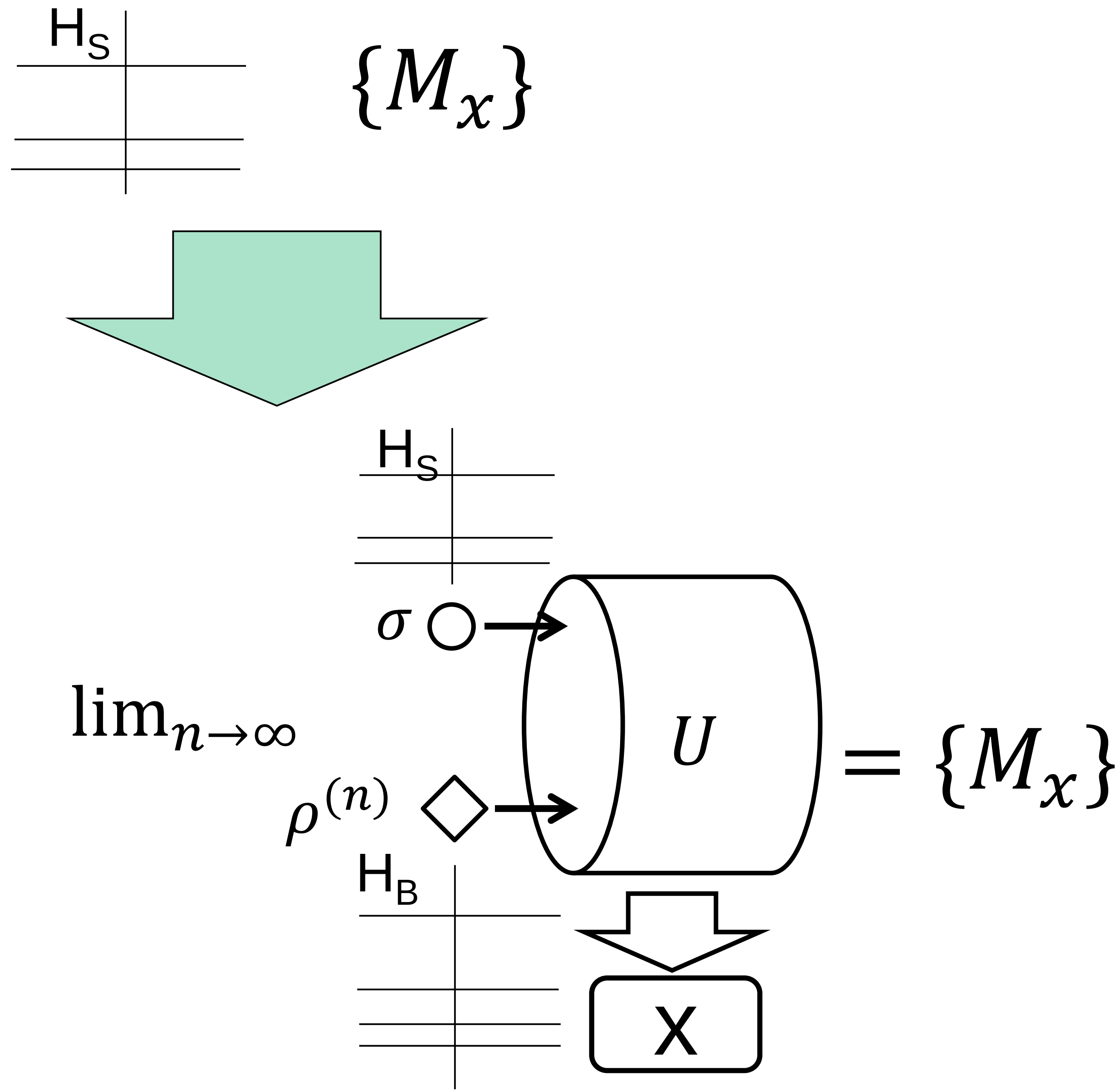


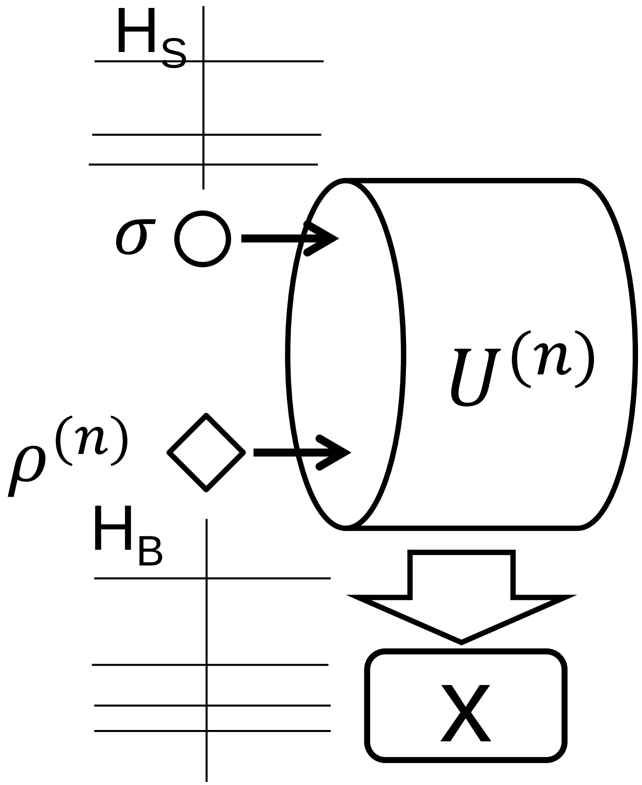
$$H_T = H_s \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B$$



How far can we go with this model?

$\exists H_B, \rho^{(n)},$
U, s. t.

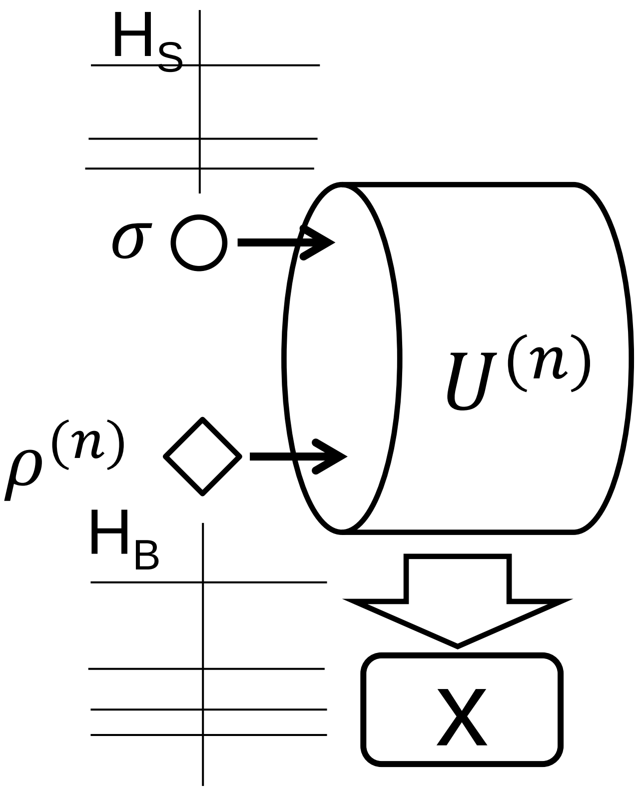




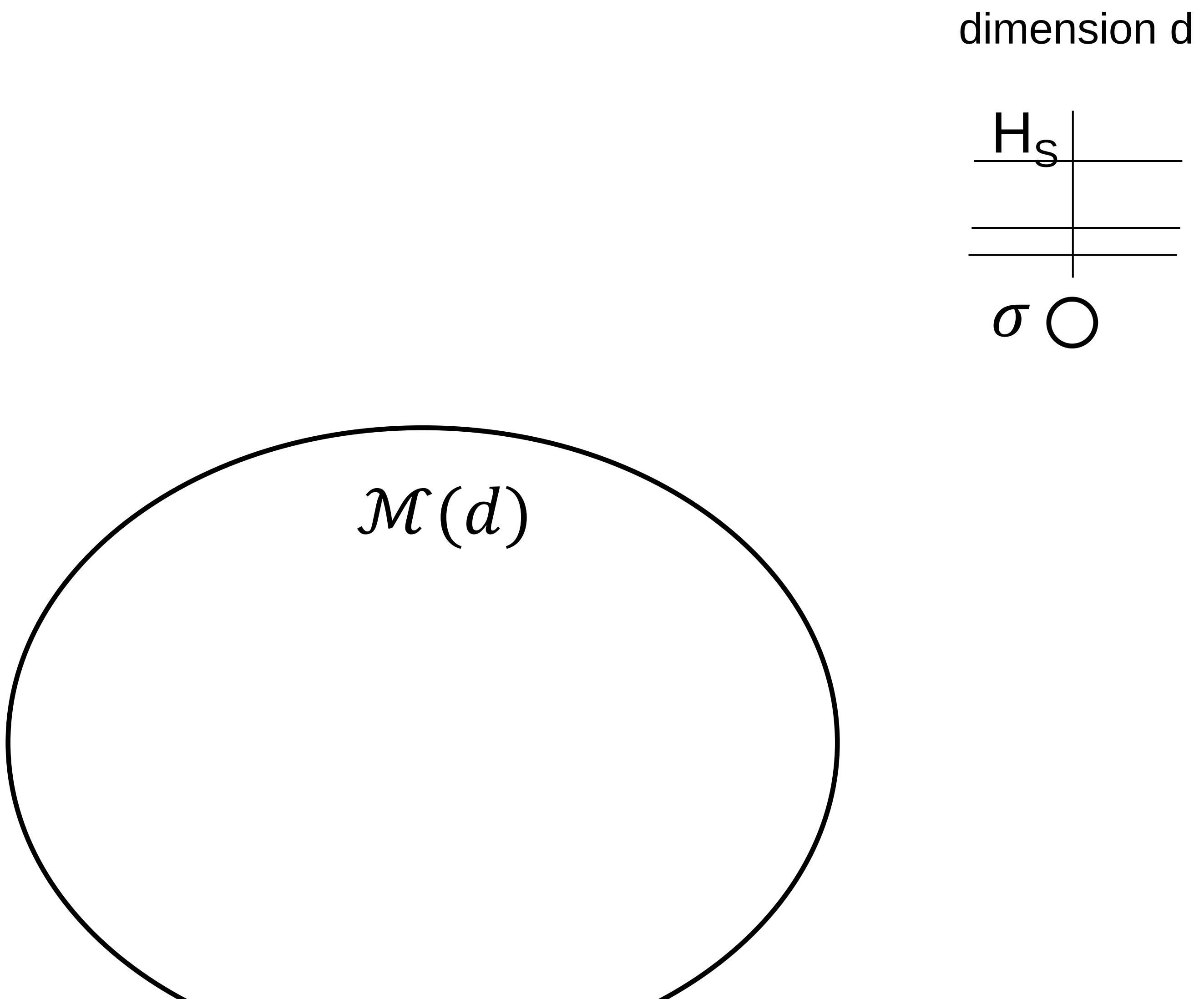
Problems

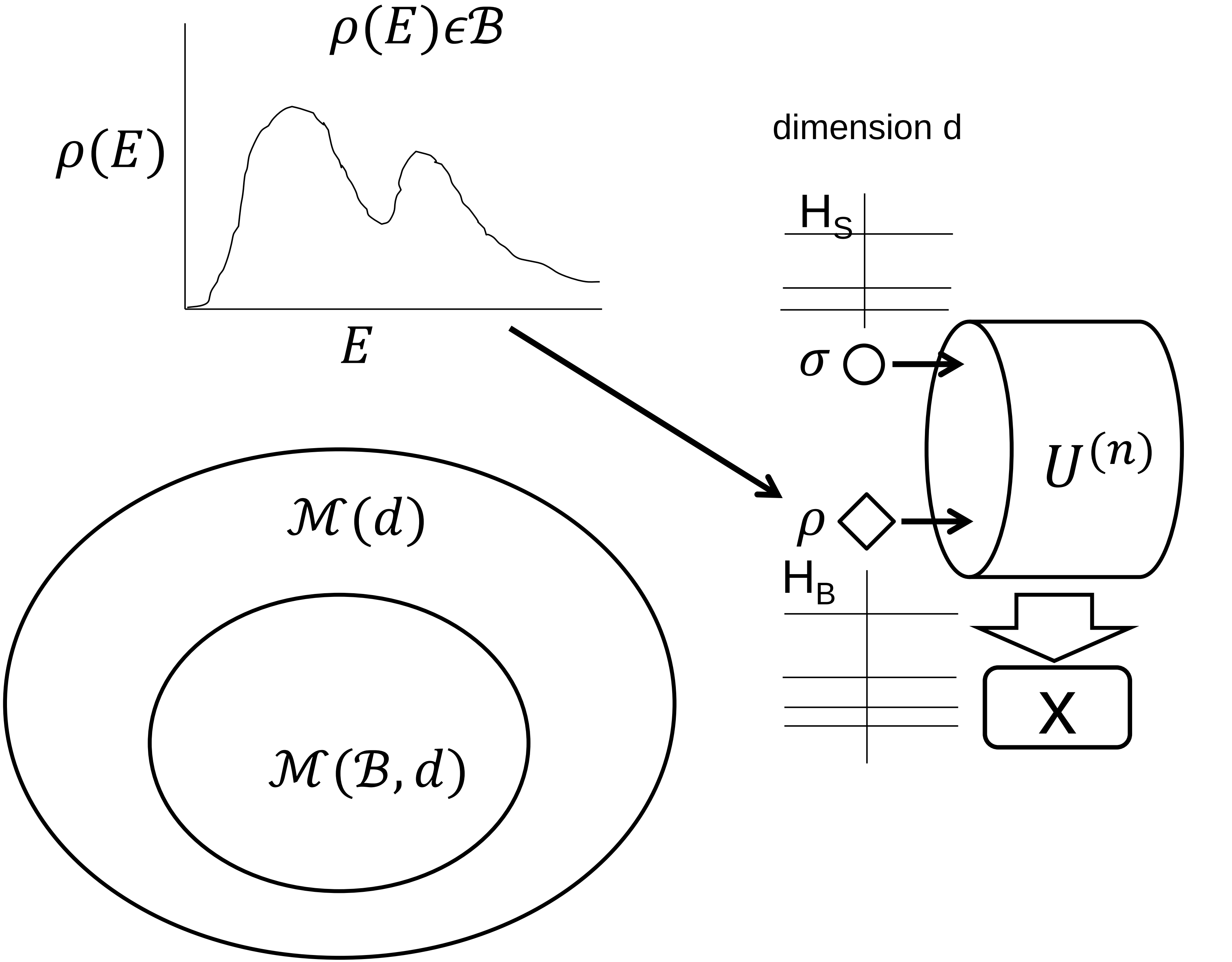
H_B, infinite dimensional

$$\lim_{n\to\infty} tr(\rho^{(n)} H_B) \to \infty$$



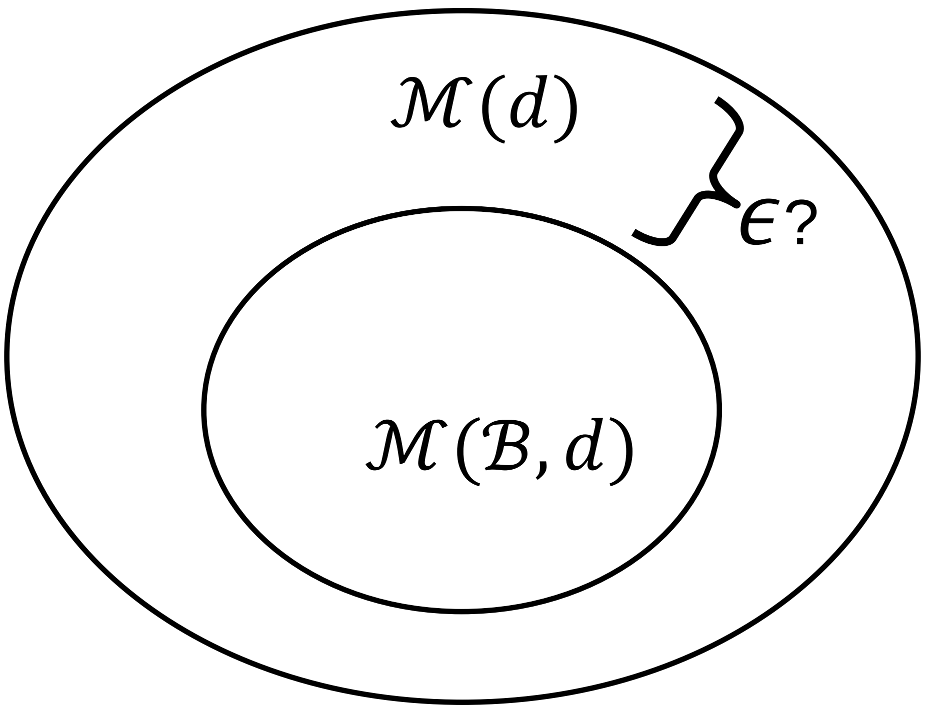
What can we measure under reasonable assumptions on the energy spectrum of the battery?


dimension d
H_S
σ
$\mathcal{M}(d)$



ρ(E)ϵℬ
ρ(E)
E
dimension d
H_S
σ
U^(n)
ρ
H_B
X
ℳ(d)
ℳ(ℬ,d)

How is a measurement device limited by the energy spectrum of its battery?



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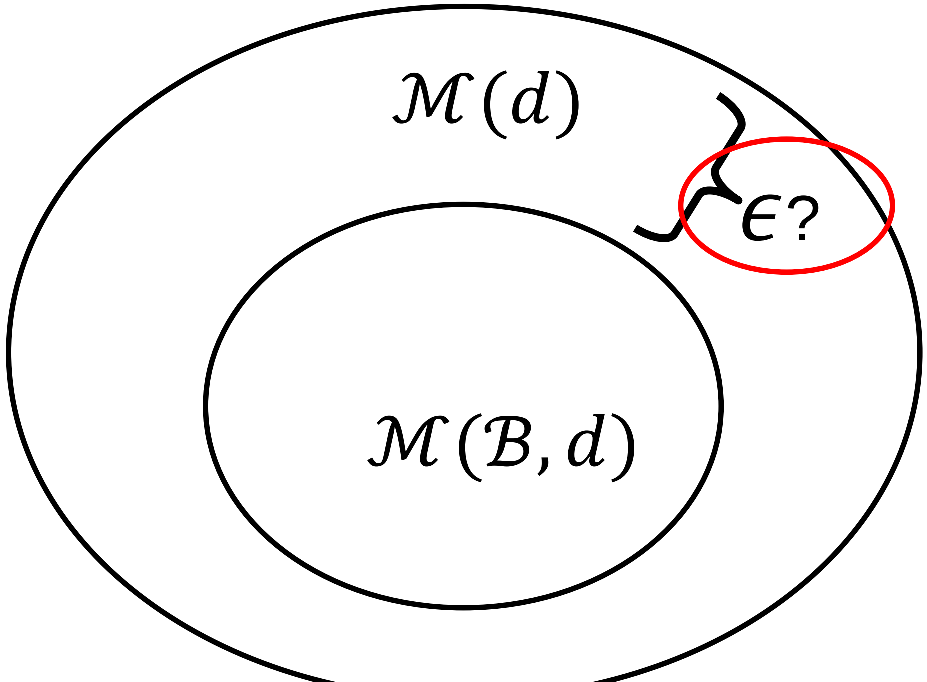
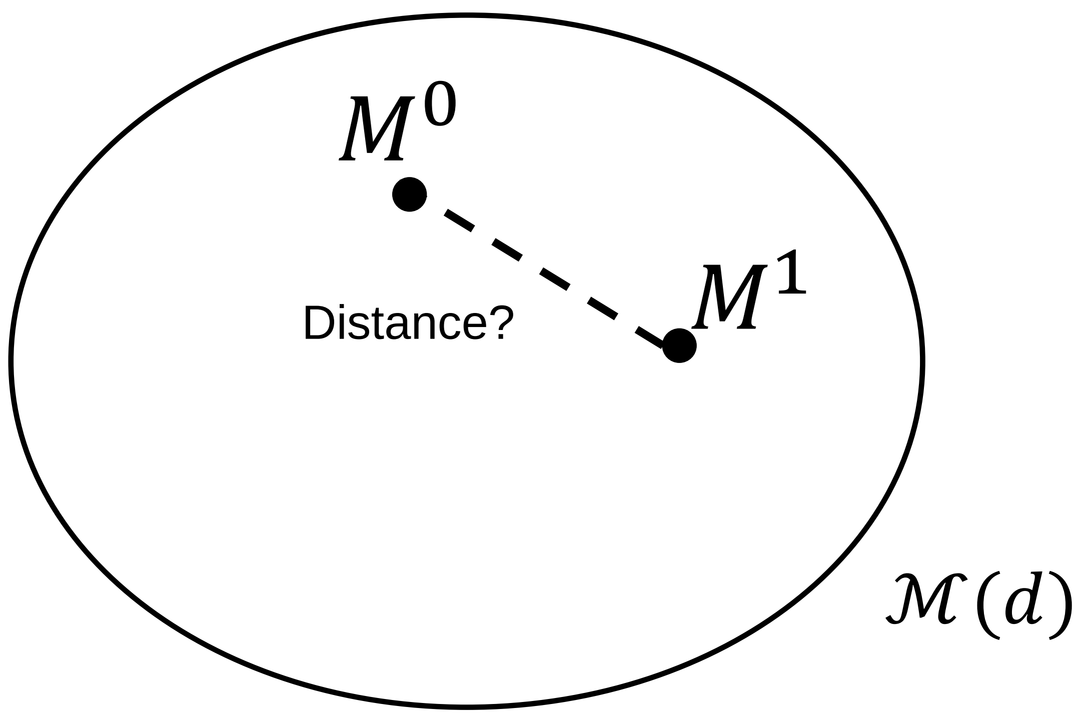
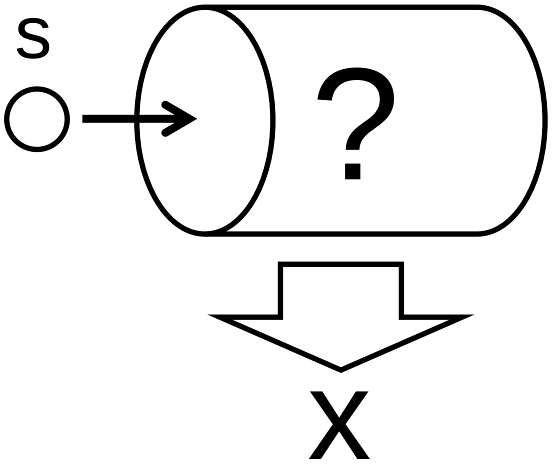


M^0, M^1 $\left(\begin{array}{c} \{M^a{}_x\}_{x=1,2,3,....} \\ M^a{}_x \geq 0, \sum_x M^a{}_x = \mathbb{I} \end{array} \right)$



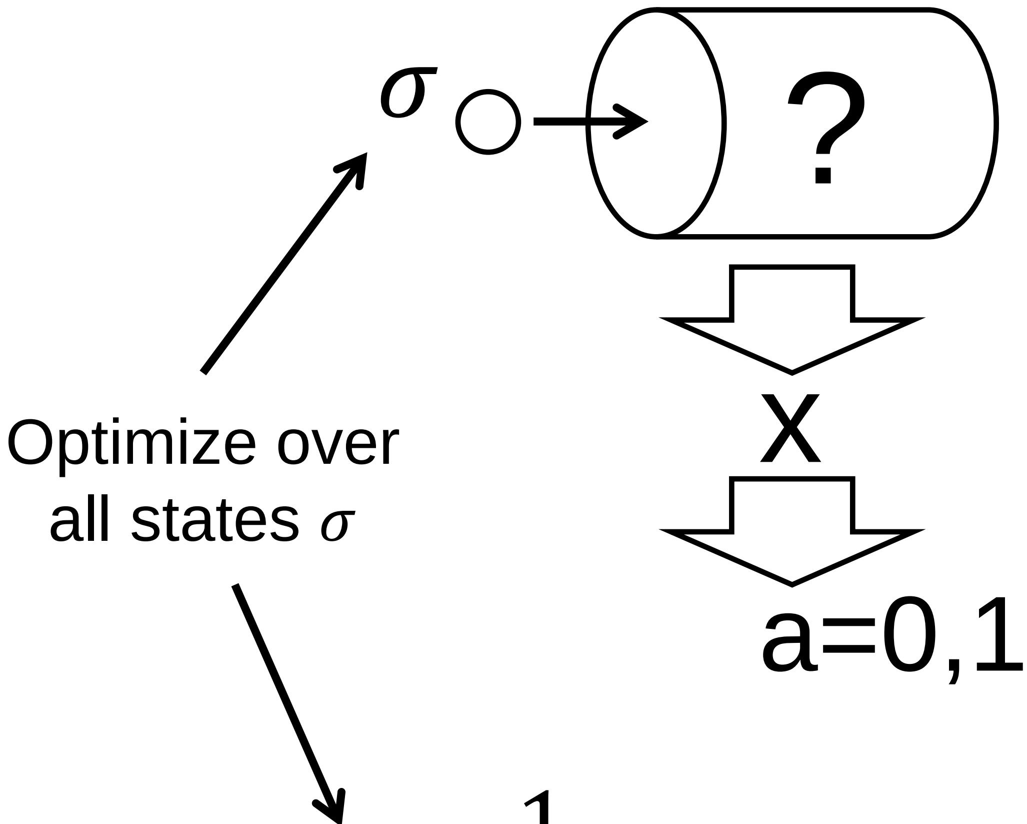


? = With probability ½, M^0_x ($a=0$)

With probability ½, M^1_x ($a=1$)

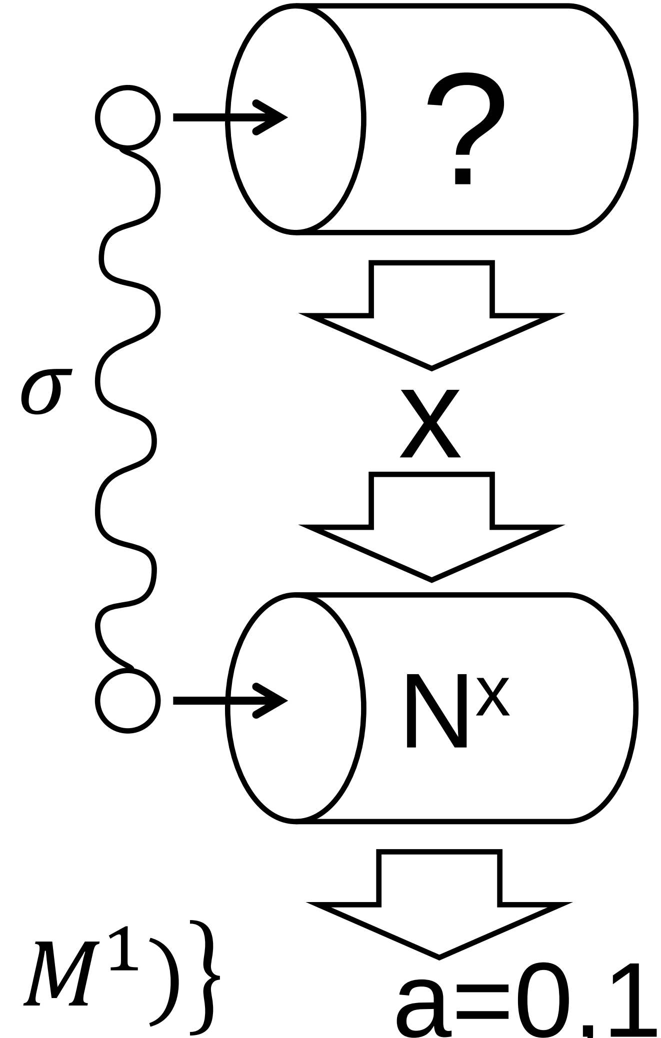
What is the value of a?

Classical Strategy



$$P_C = \frac{1}{2}\{1 + dist_C(M^0, M^1)\}$$

Quantum Strategy



Optimize over all states σ

$$P_Q = \frac{1}{2}\{1 + dist_Q(M^0, M^1)\}$$

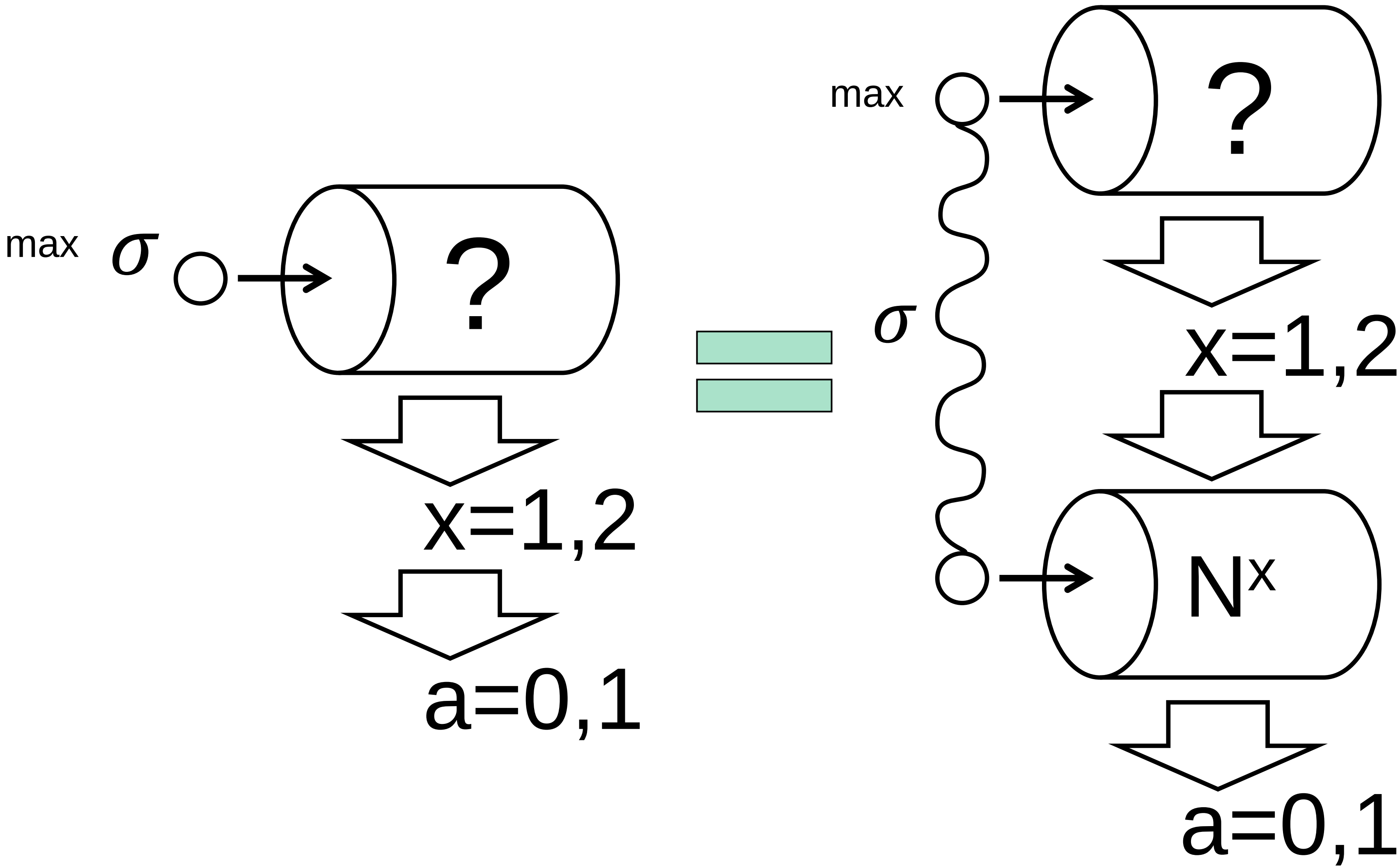
Trivia

$dist_Q(M^0, M^1), dist_C(M^0, M^1)$, distances

Trivia

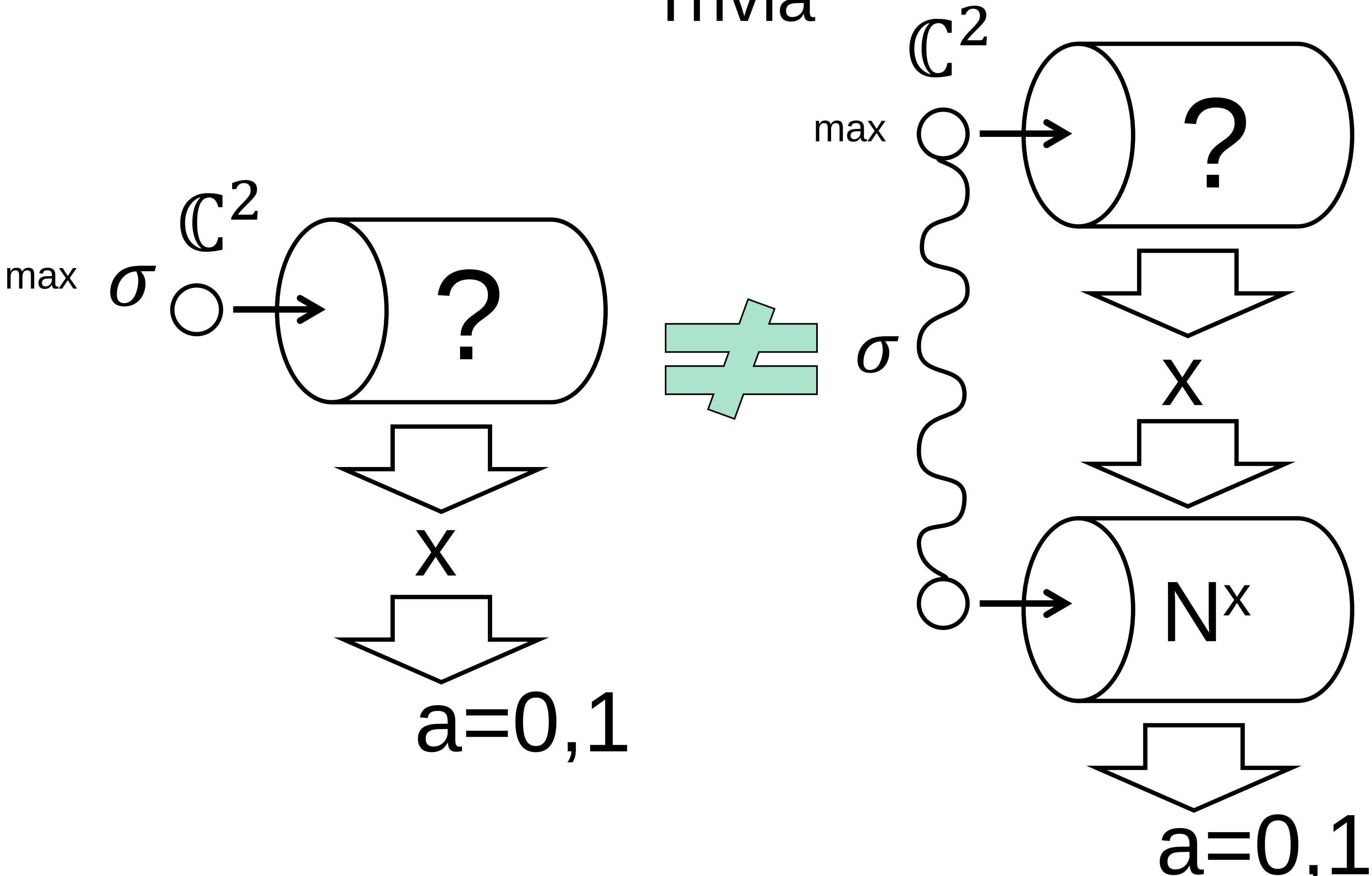
$$1 \geq dist_Q(M^0, M^1) \geq dist_C(M^0, M^1) \geq 0$$

Trivia

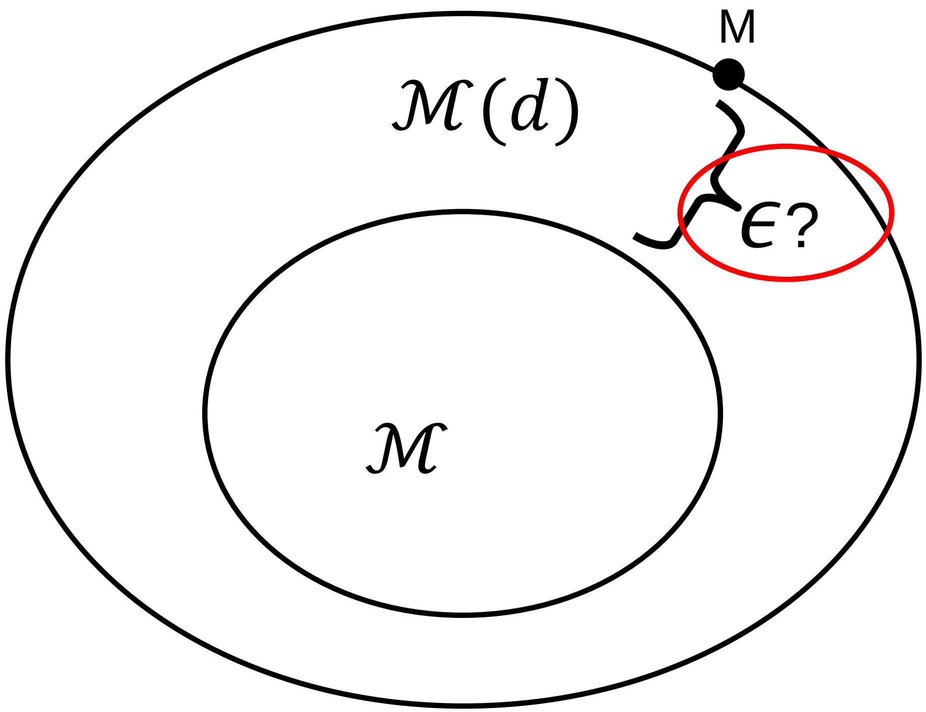


$$dist_C(M^0, M^1) = dist_Q(M^0, M^1)$$

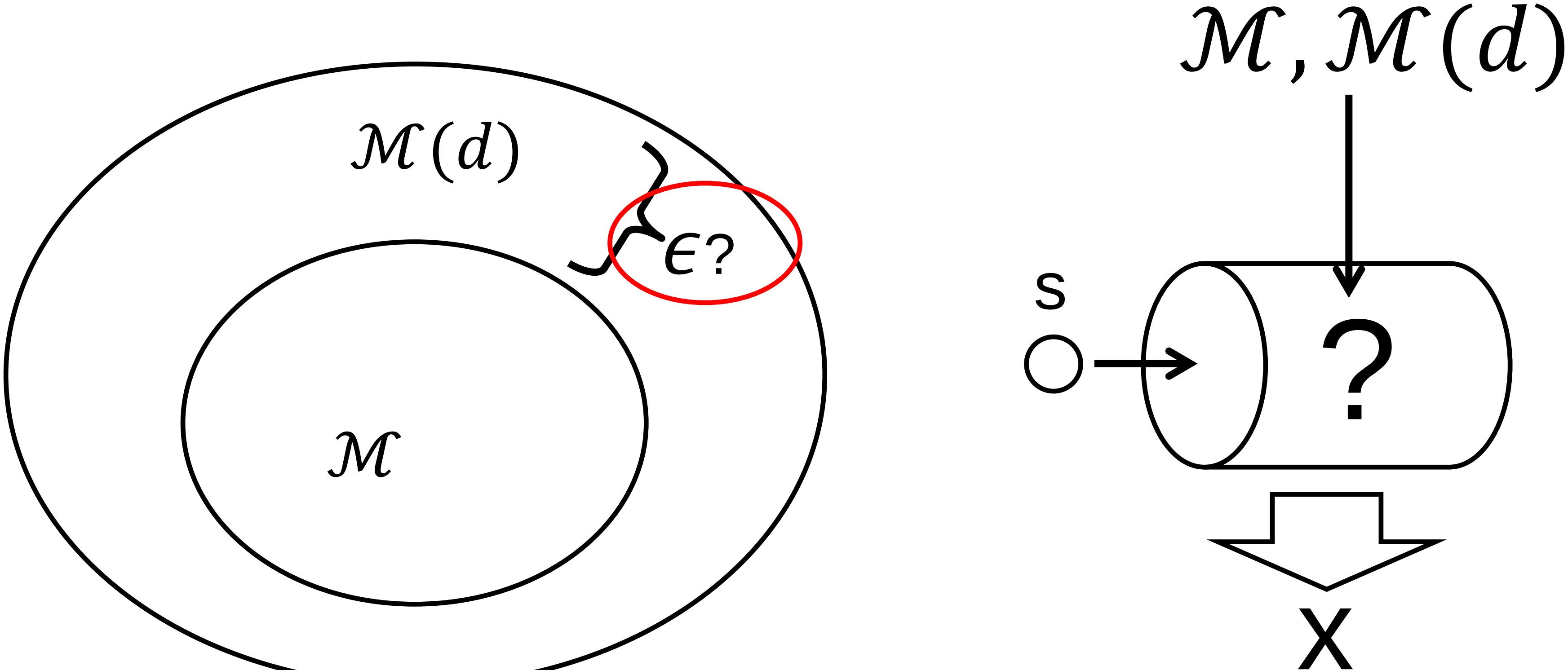
Trivia



$$dist_C(M^0, M^1) \neq dist_Q(M^0, M^1)$$

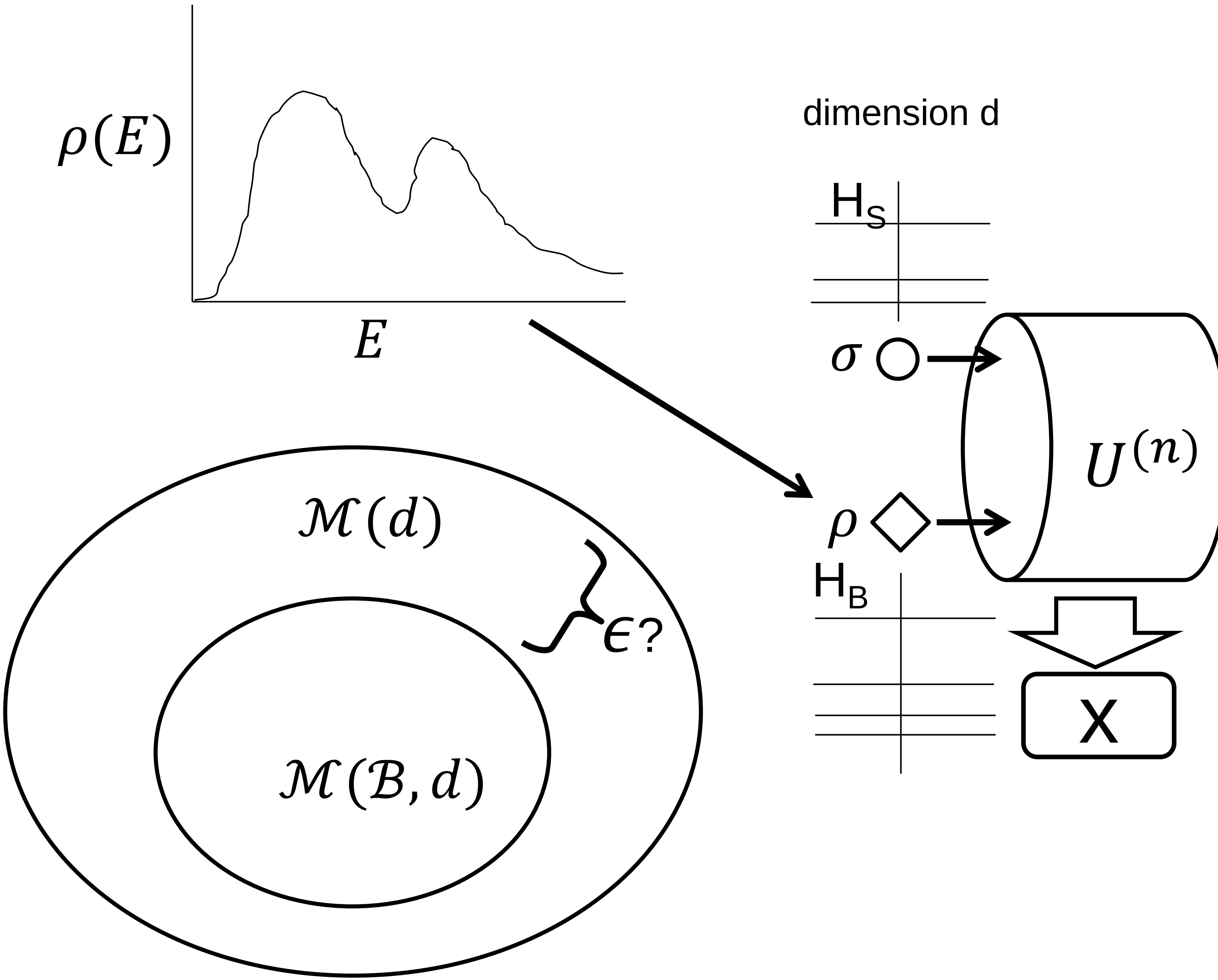


$$\epsilon_{C,Q} = \max\{dist_{C,Q}(M, \mathcal{M}) : M \in \mathcal{M}(d)\}$$



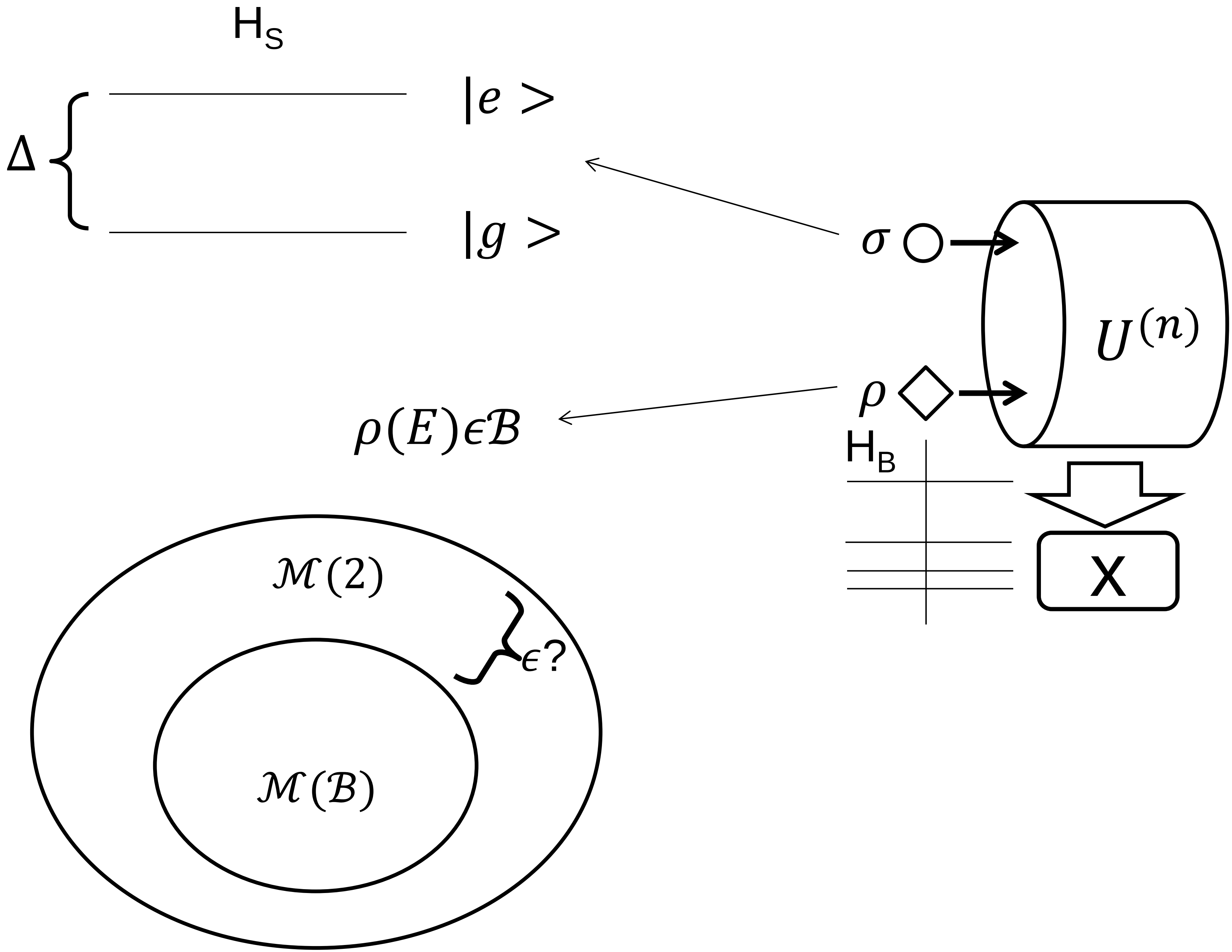
$$\epsilon_{C,Q} = \max\{dist_{C,Q}(M, \mathcal{M}) \colon M \in \mathcal{M}(d)\}$$


Worth1000.com

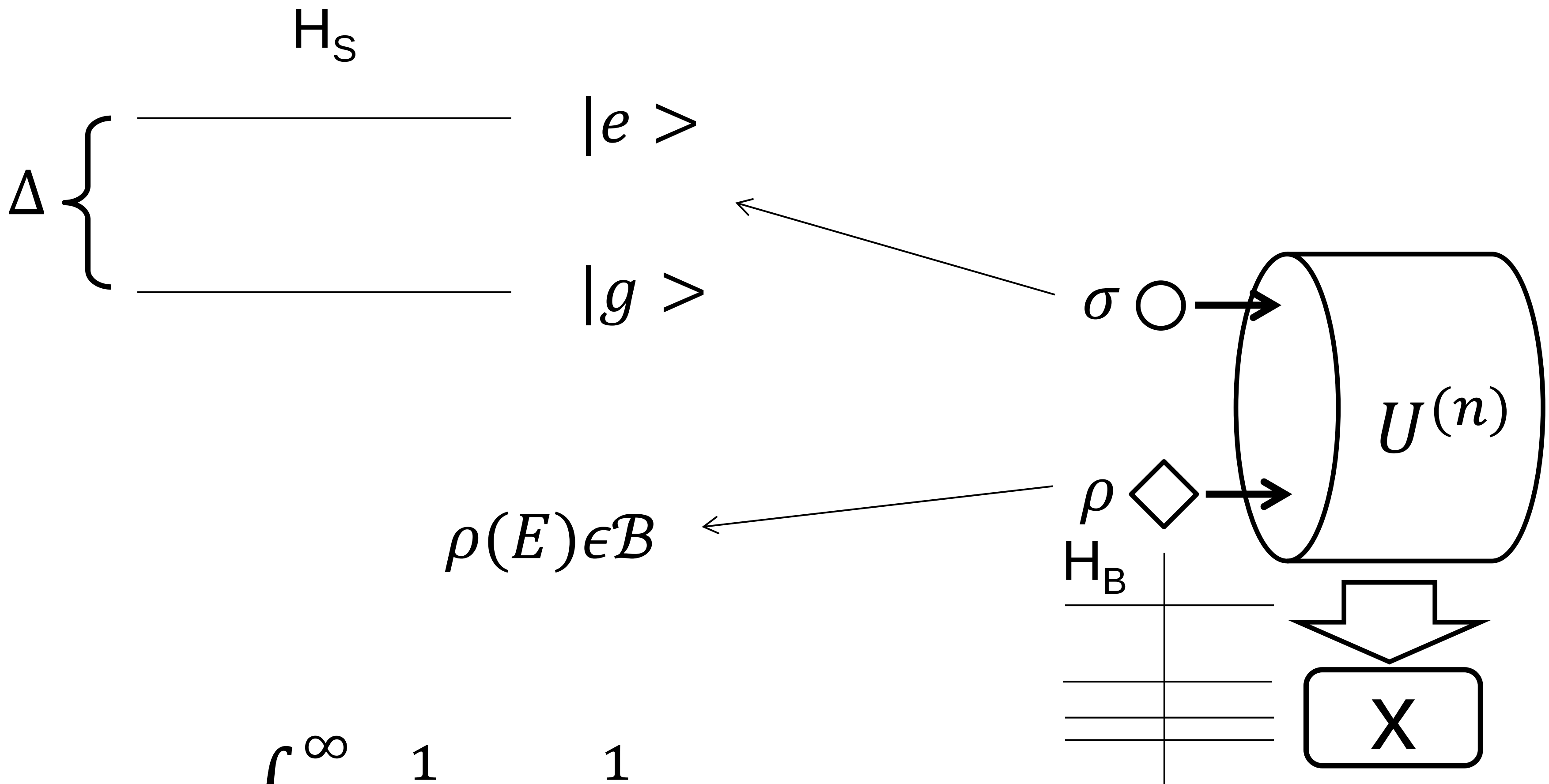


$\rho(E)$
E
dimension d
H_S
σ
ρ
H_B
$U^{(n)}$
X
$\mathcal{M}(d)$
ϵ?
$\mathcal{M}(\mathcal{B}, d)$

The qubit case

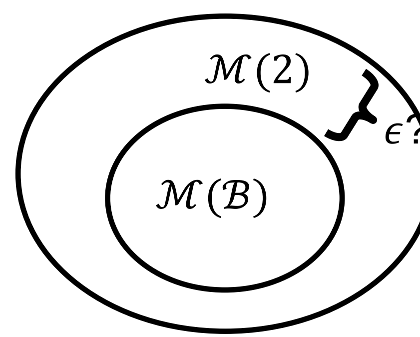


H_S
Δ
$|e>$
$|g>$
σ
$U^{(n)}$
ρ
$\rho(E)\epsilon\mathcal{B}$
H_B
X
$\mathcal{M}(2)$
ϵ?
$\mathcal{M}(\mathcal{B})$



$$\tau = \max_{\varrho\,\in\,\mathcal{B}} \int_0^\infty \rho^{\frac{1}{2}}(E)\rho^{\frac{1}{2}}(E+\Delta)dE$$

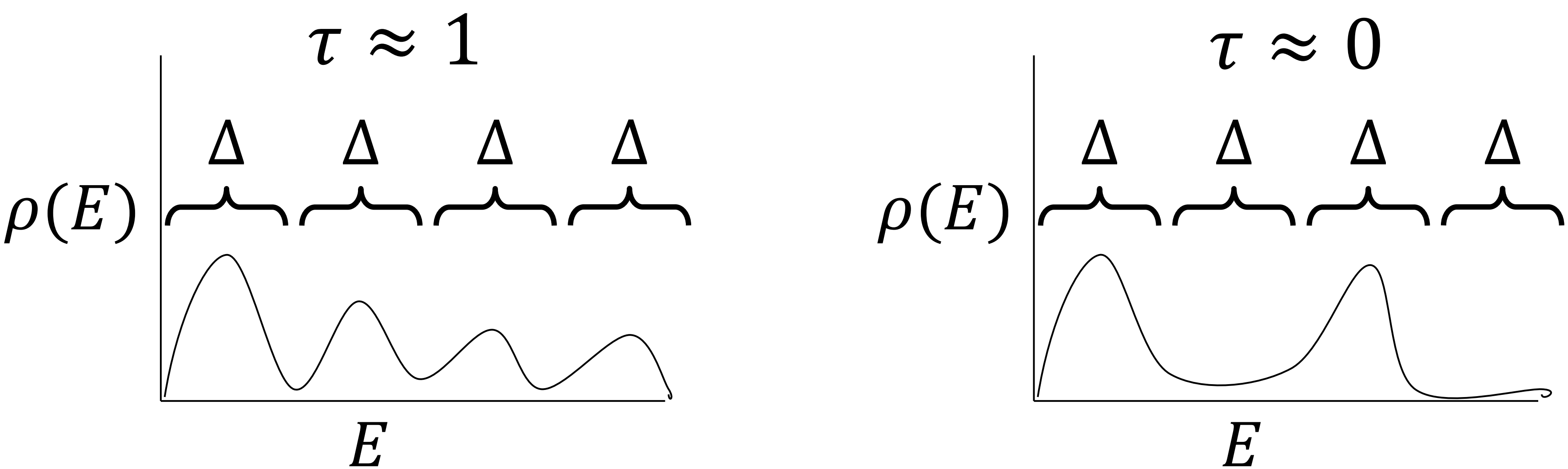
$$\epsilon_C = \epsilon_Q = \frac{1}{2}(1-\tau)$$



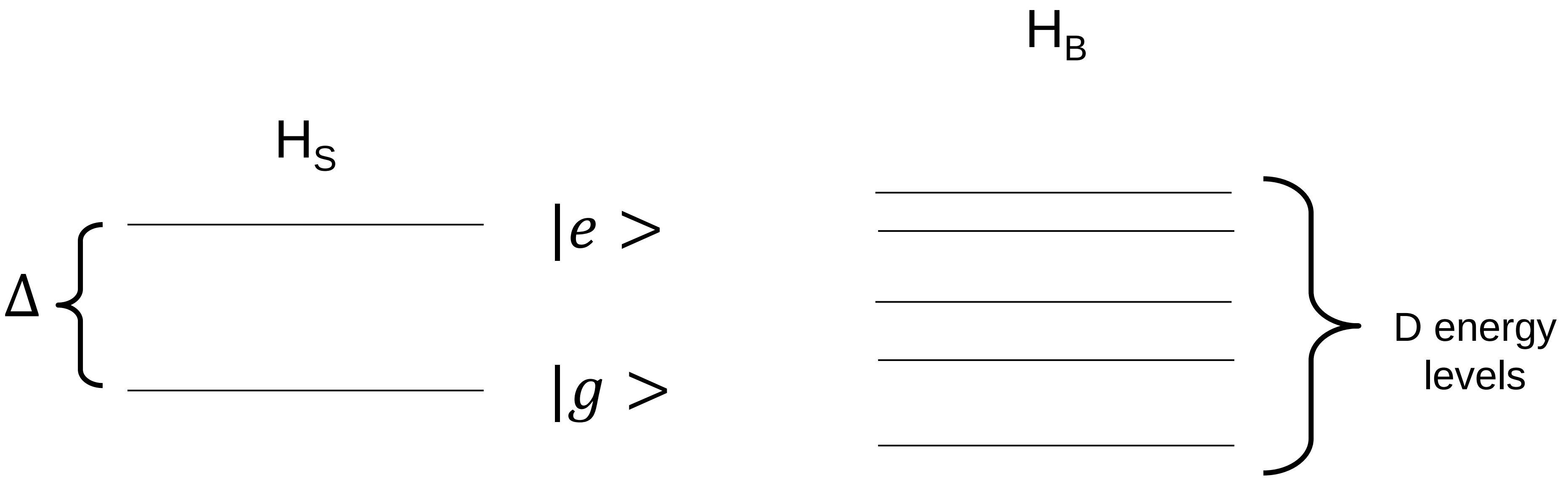
$$\tau = \max \int_0^\infty \rho^{\frac{1}{2}}(E)\rho^{\frac{1}{2}}(E + \Delta)dE$$

“The closer to 1, the more we can measure”

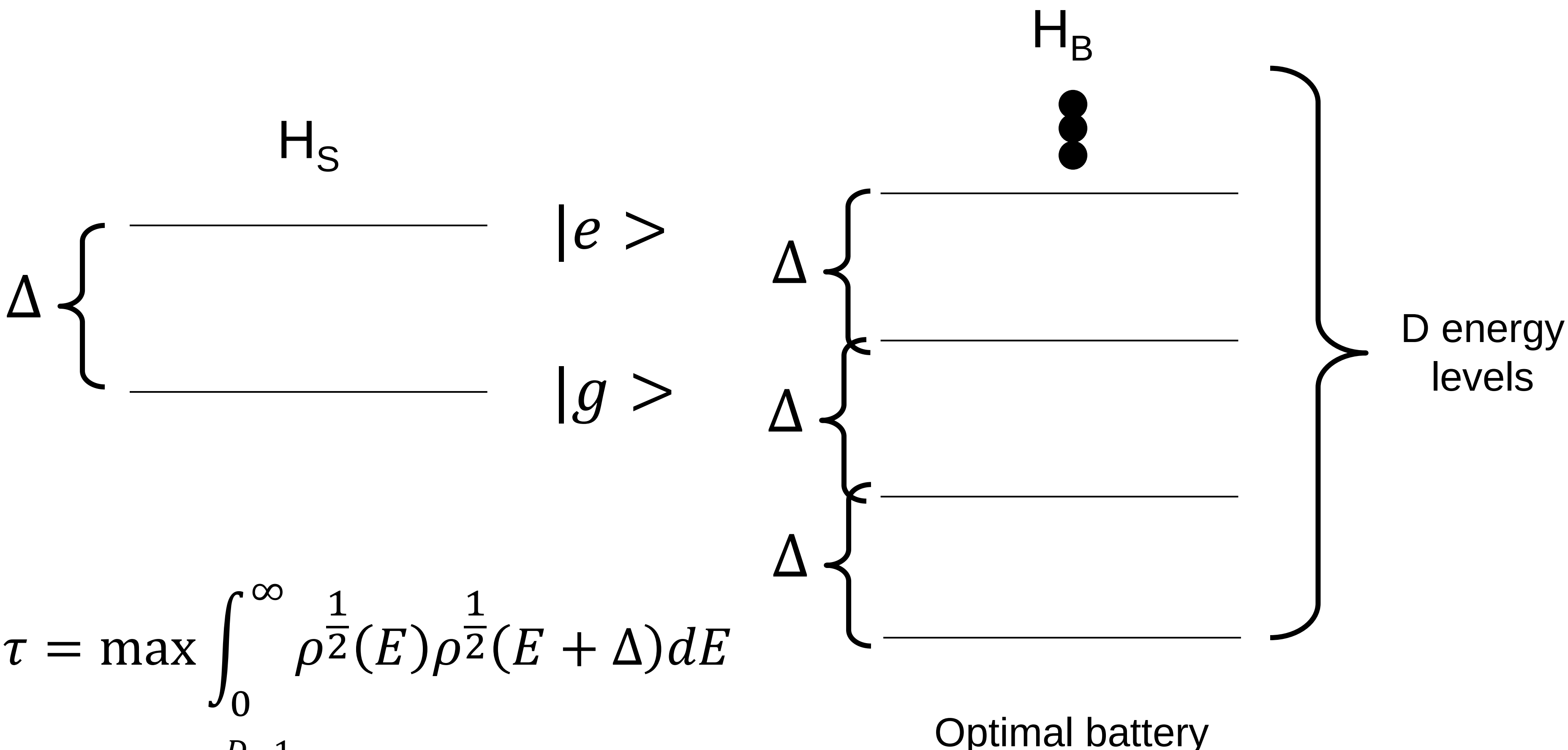
$$\tau = \max \int_0^\infty \rho^{\frac{1}{2}}(E)\rho^{\frac{1}{2}}(E+\Delta)dE$$



Case of interest: battery with finitely many energy levels



Case of interest: battery with finitely many energy levels

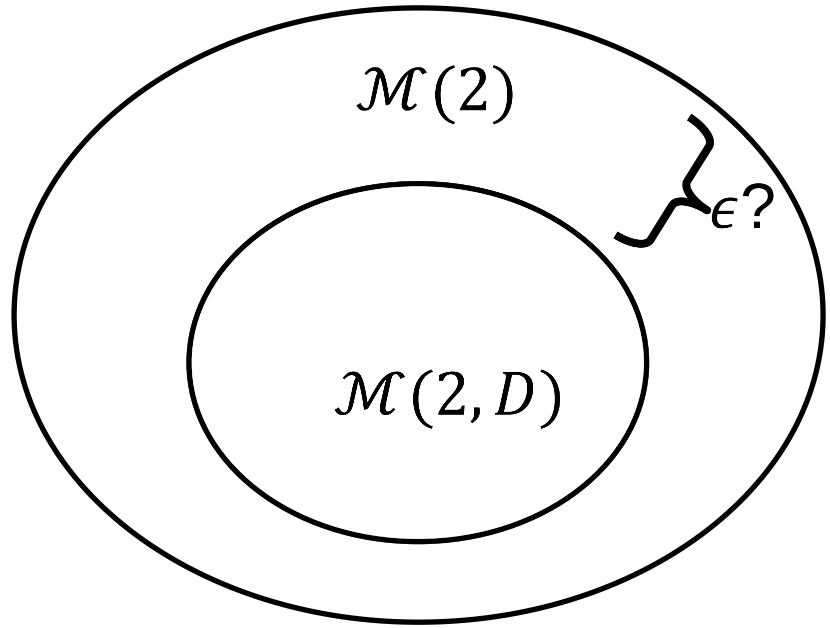


$$\tau = \max \int_0^\infty \rho^{\frac{1}{2}}(E)\rho^{\frac{1}{2}}(E+\Delta)dE$$

$$\varrho(E) = \sum_{k=0}^{D-1} p_k \delta(E - \Delta k)$$

Case of interest: battery with finitely many energy levels

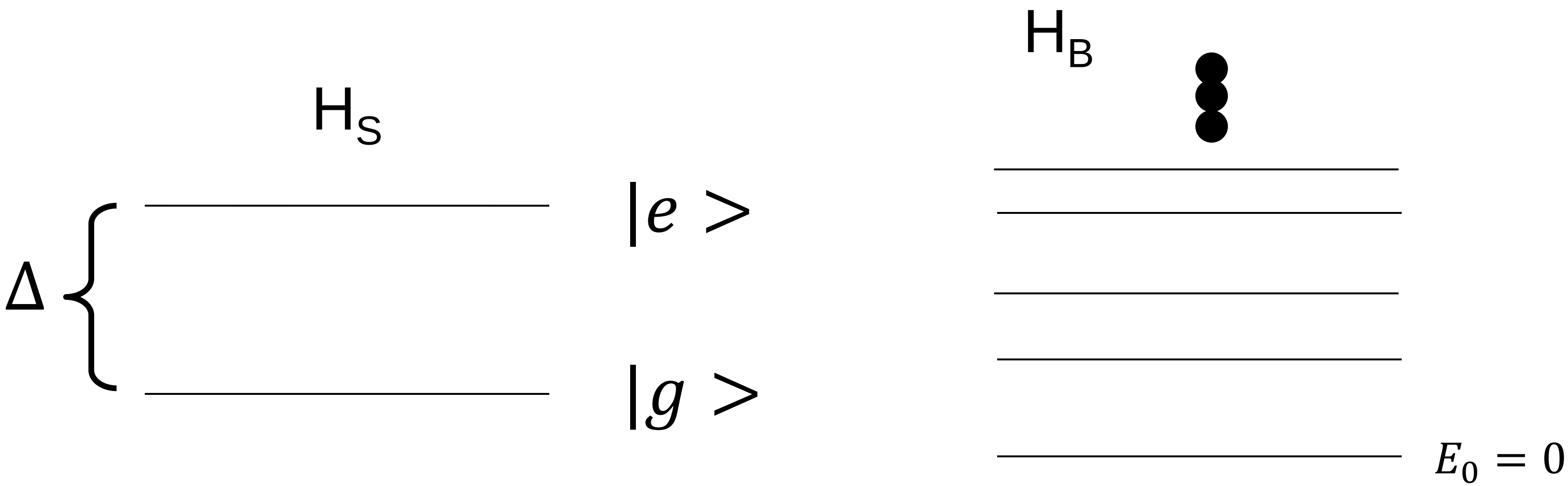
$$\tau = \cos\left(\frac{\pi}{D+1}\right)$$



$$\varepsilon_{C,Q} = \frac{1}{2}\left\{1 - \cos\left(\frac{\pi}{D+1}\right)\right\} \approx O\left(\frac{1}{D^2}\right)$$

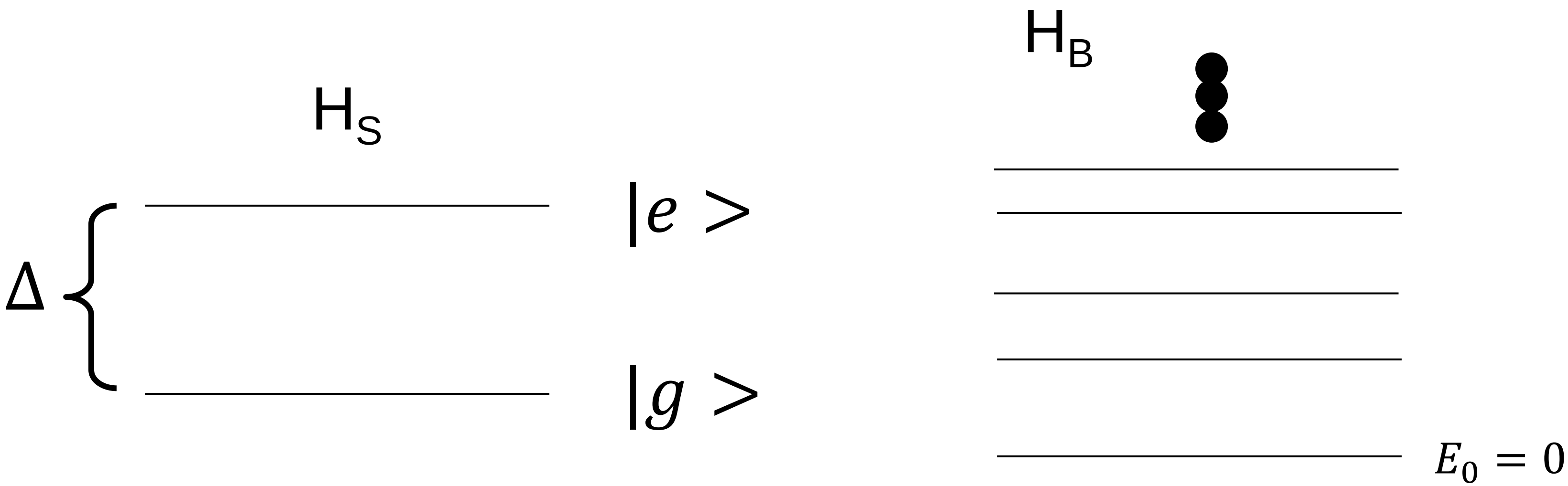

YES?
YOU CALLED?

Case of interest: battery with finite average energy



$$\int_0^{\infty} \varrho(E) E dE \leq \bar{E}$$

Case of interest: battery with finite average energy



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$$\int_0^\infty \varrho(E)E\,dE \leq \bar{E}$$

Case of interest: battery with finite average energy

$$\tau = \varphi\left(\frac{\bar{E}}{\Delta}\right)$$

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$j_{n,1}$ ≡1st positive zero of $J_n(x)$

Case of interest: battery with finite average energy

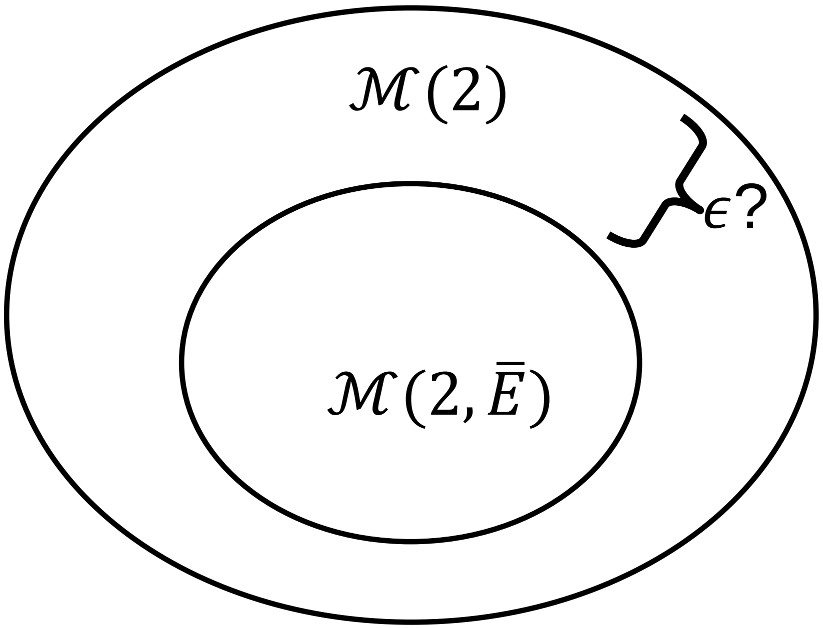
$$\tau = \varphi\left(\frac{\bar{E}}{\Delta}\right)$$

$$\varphi(z) = min_{\lambda \geq 0} \frac{z + \mu(\lambda)}{2\lambda}$$

$$\varphi(z) \approx 1 - \frac{0.9468}{z^2}$$

$$z \gg 1$$

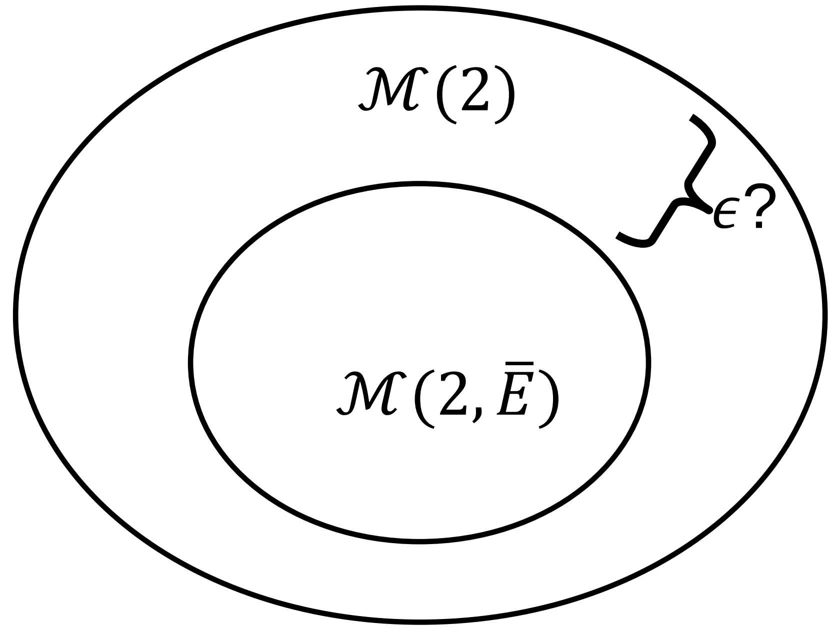
Case of interest: battery with finite average energy



$$\varepsilon_{C,Q} = \frac{1}{2}\left\{1 - \varphi\left(\frac{\bar{E}}{\Delta}\right)\right\} \approx \frac{0.4734\Delta^2}{\bar{E}^2}$$

$$\bar{E} \gg \Delta$$

Case of interest: battery with finite average energy



$$\tau = \max \int_0^\infty \rho^{\frac{1}{2}}(E)\rho^{\frac{1}{2}}(E+\Delta)dE$$

$$\int_0^\infty \varrho(E)EdE \leq \bar{E}$$

Optimal states

Power states

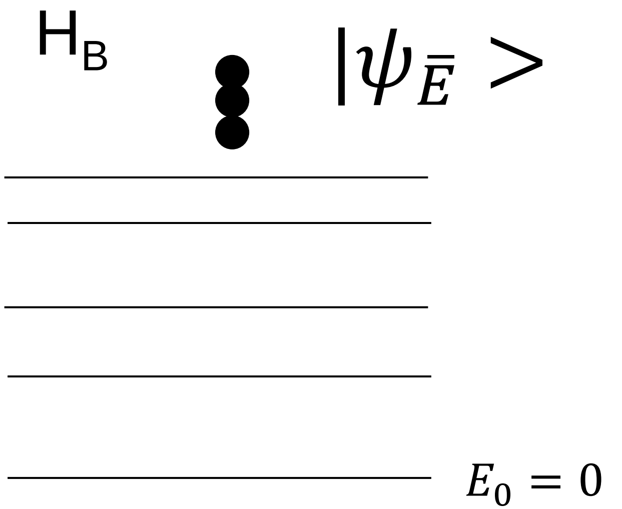
$$\rho^* = |\psi_{\bar{E}} >< \psi_{\bar{E}}|$$

Case of interest: battery with finite average energy

Power states

$$|\psi_{\bar{E}}> = \sum_{k=0}^{\infty} c_k |k>$$

$$H_B|k> = \Delta k|k>$$



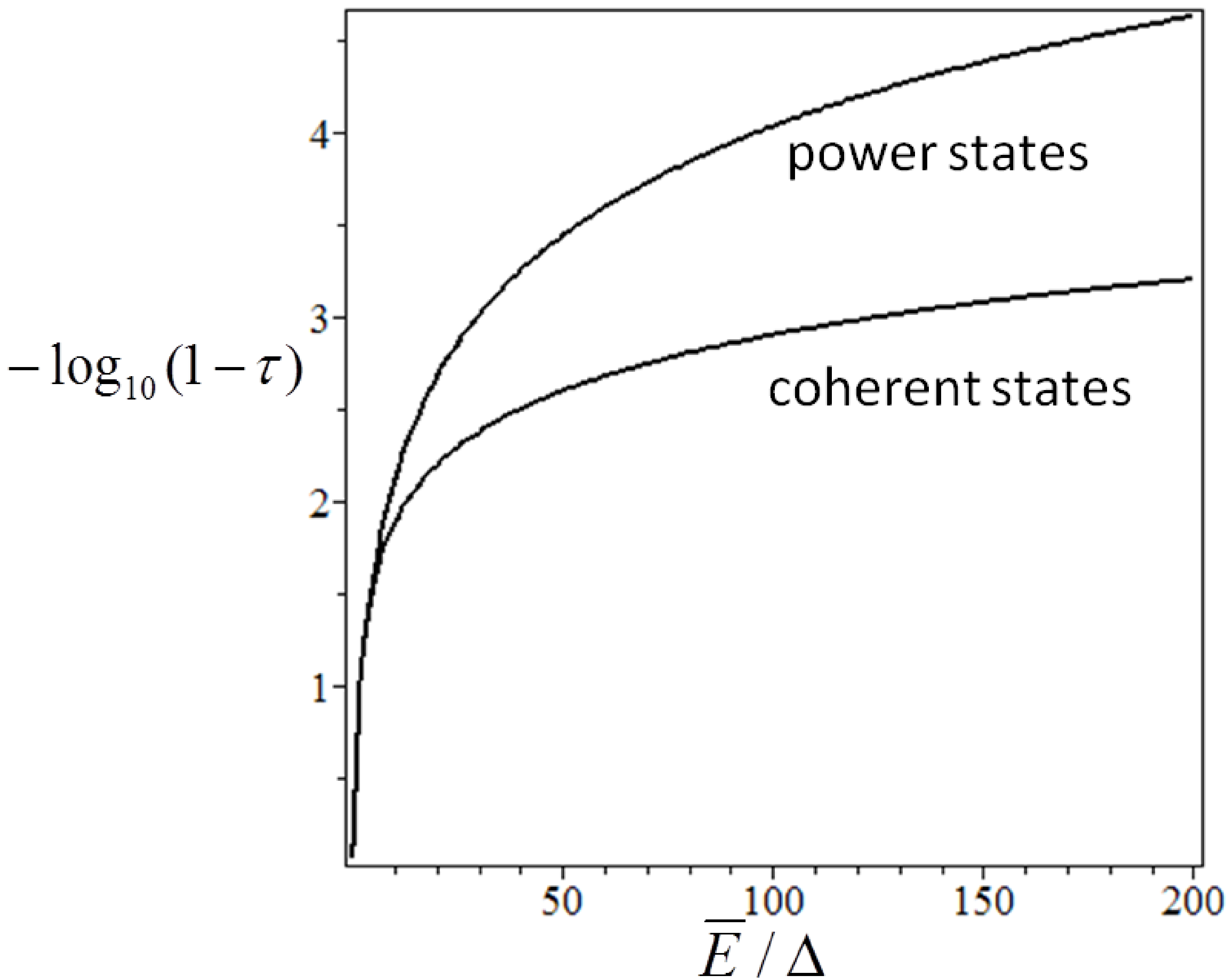
$$c_{k+1} = \frac{k + \mu(\lambda^*)}{\lambda^*} c_k - c_{k-1}$$

Comparison with coherent states

$$|\psi_{\bar{E}} > = \sum_{k=0}^{\infty} c_k |k > \qquad \tau \approx 1 - \frac{0.9468\Delta^2}{\bar{E}^2}$$

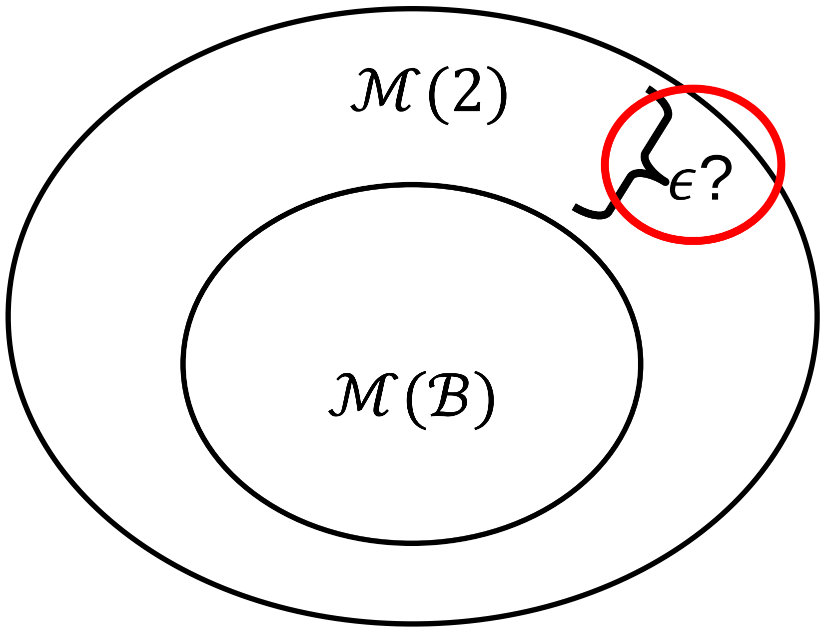
$$|\alpha > = e^{-|\alpha|^2/2} \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} |k > \qquad \tau \approx 1 - \frac{\Delta}{8\bar{E}}$$

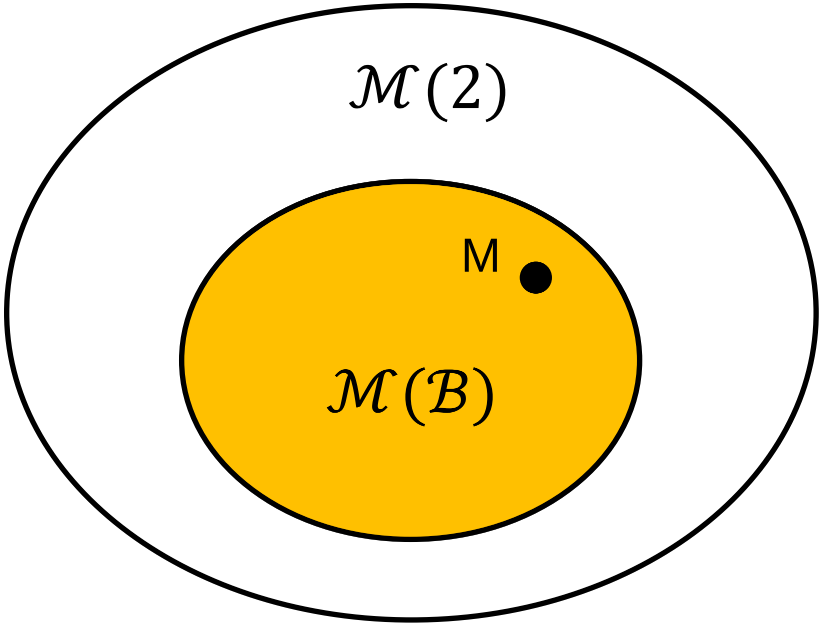
$$|\alpha|^2 = \bar{E}$$



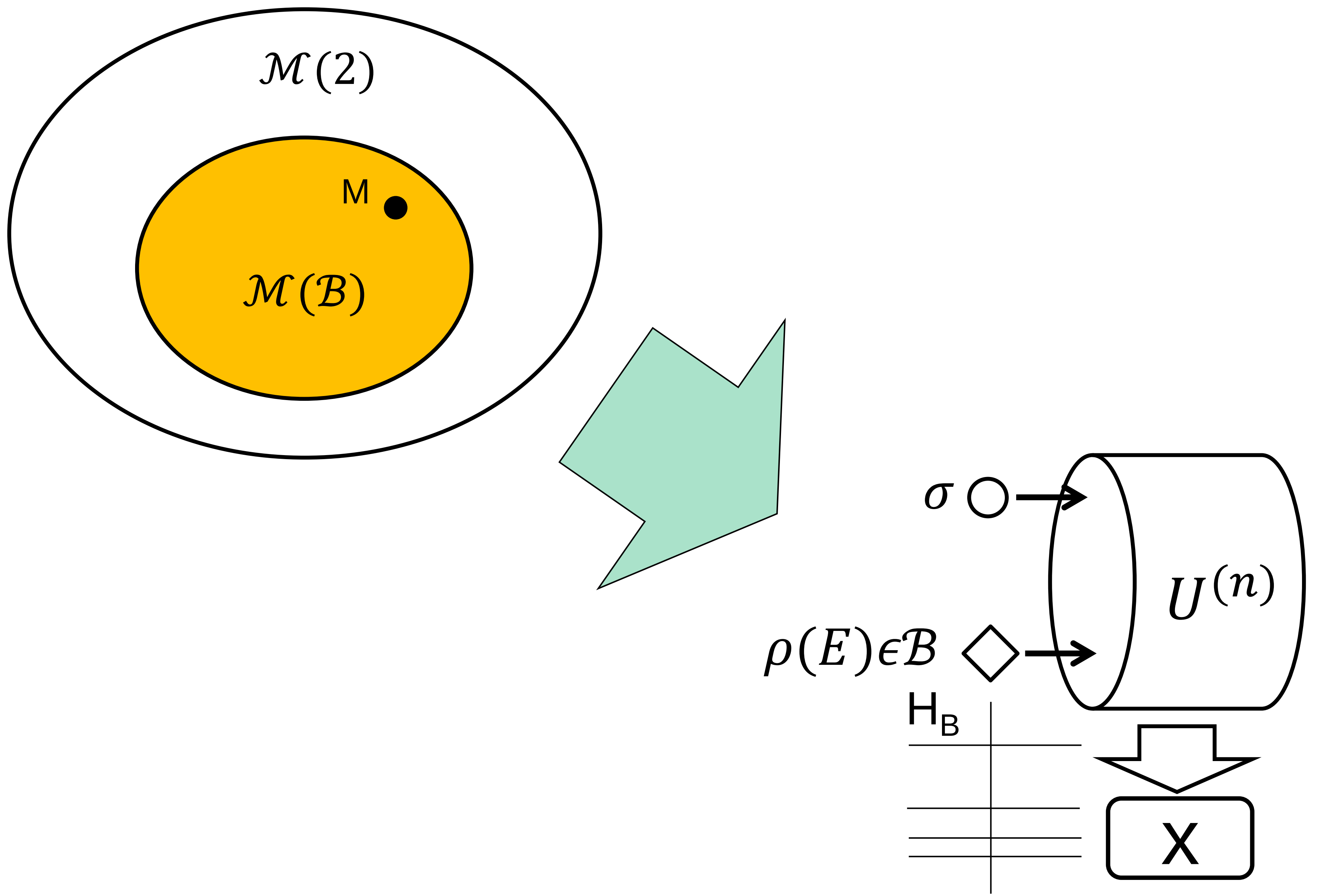
4
3
2
1
$-\log_{10}(1-\tau)$
power states
coherent states
50
100
150
200
$\overline{E}/\Delta$

Characterizations

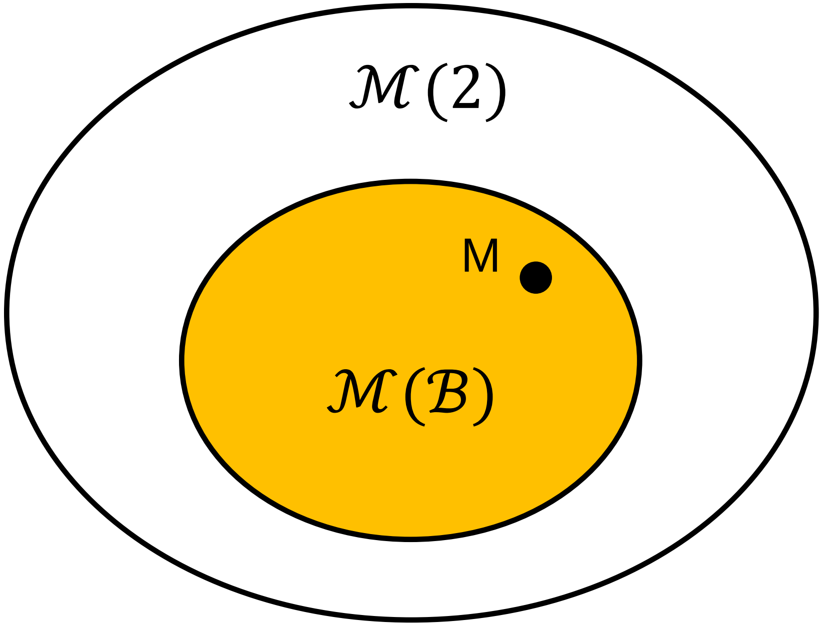

$\mathcal{M}(2)$
ϵ?
$\mathcal{M}(\mathcal{B})$



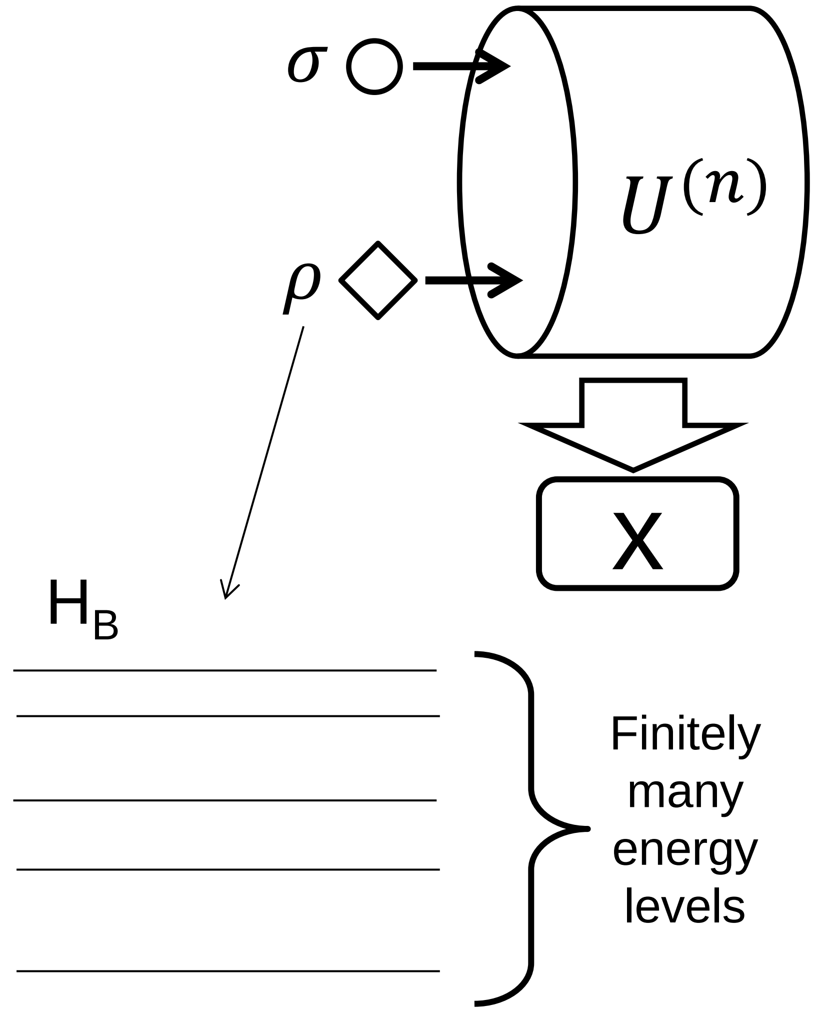
Can I realize M with the battery restriction $\mathcal{B}$?

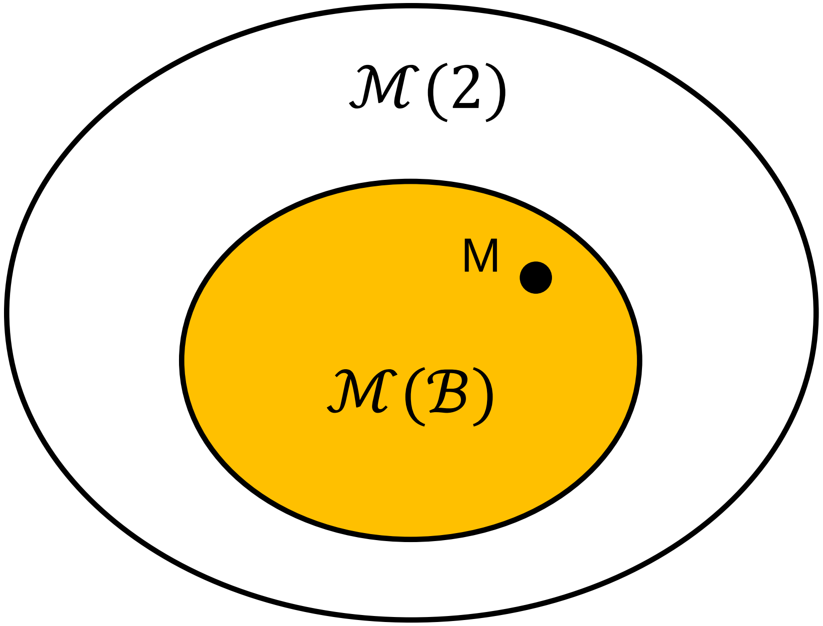


$\mathcal{M}(2)$
M
$\mathcal{M}(\mathcal{B})$
σ
$U^{(n)}$
$\rho(E) \epsilon \mathcal{B}$
H_B
X

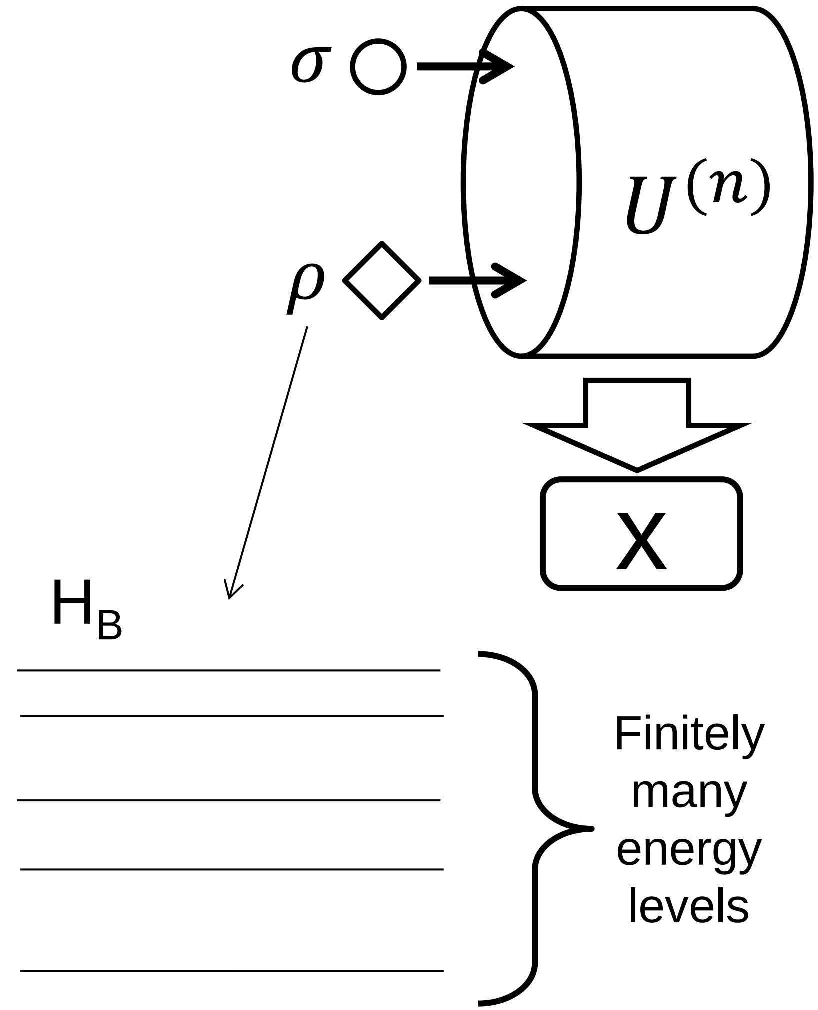


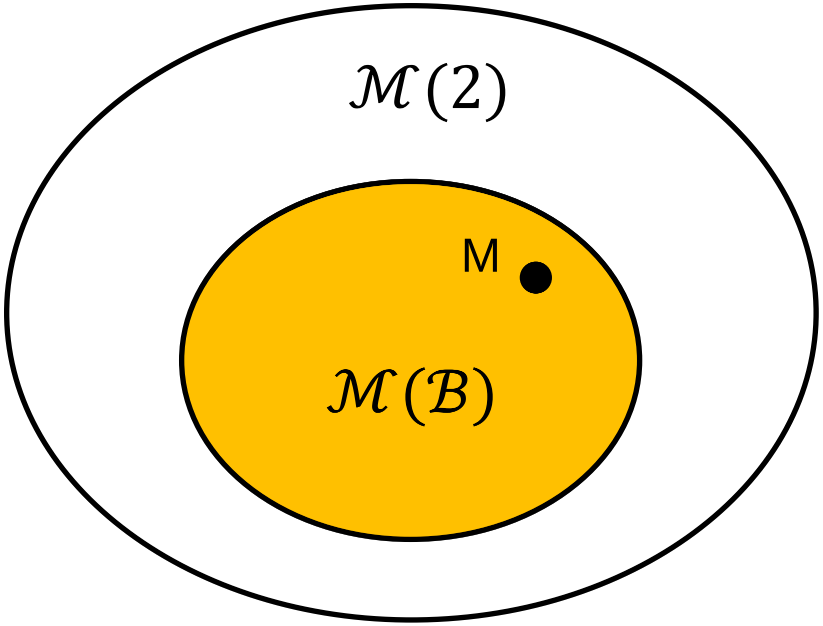
The membership problem can be decided by a single semidefinite program (SDP).



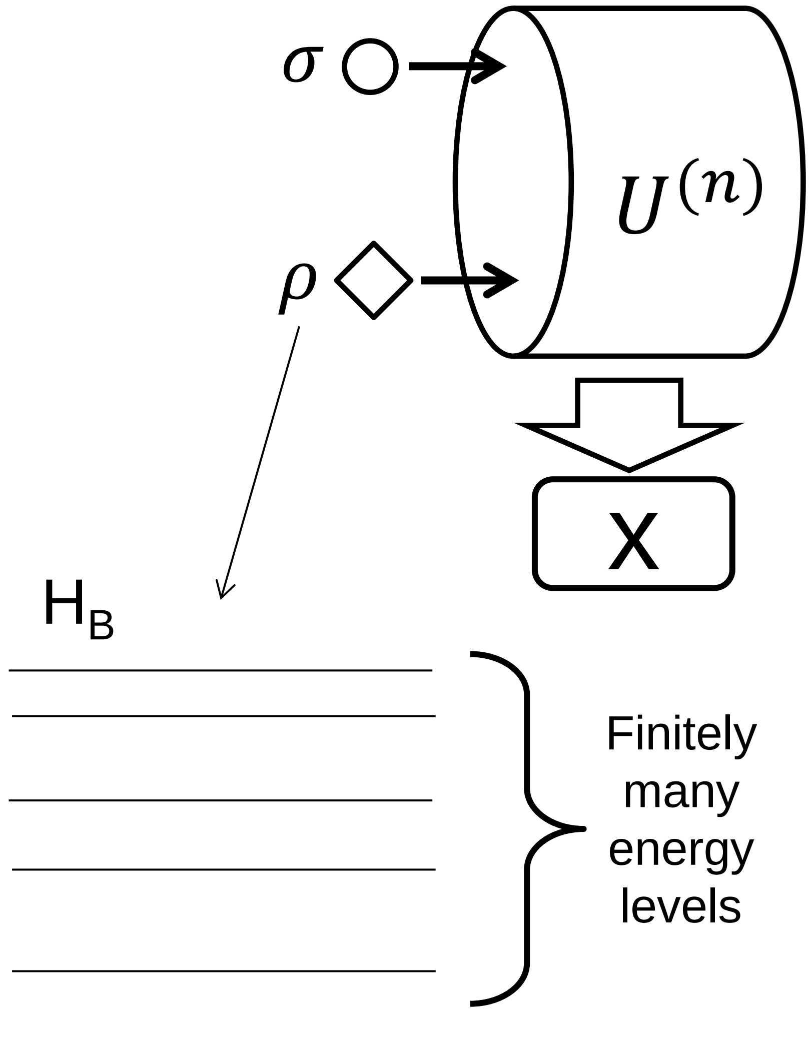


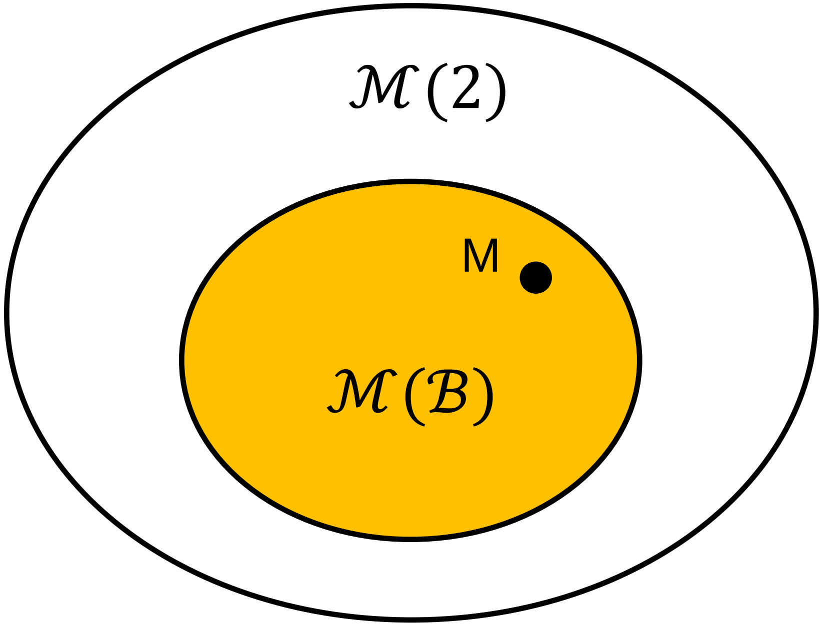
Our algorithm also returns an implementation of M.

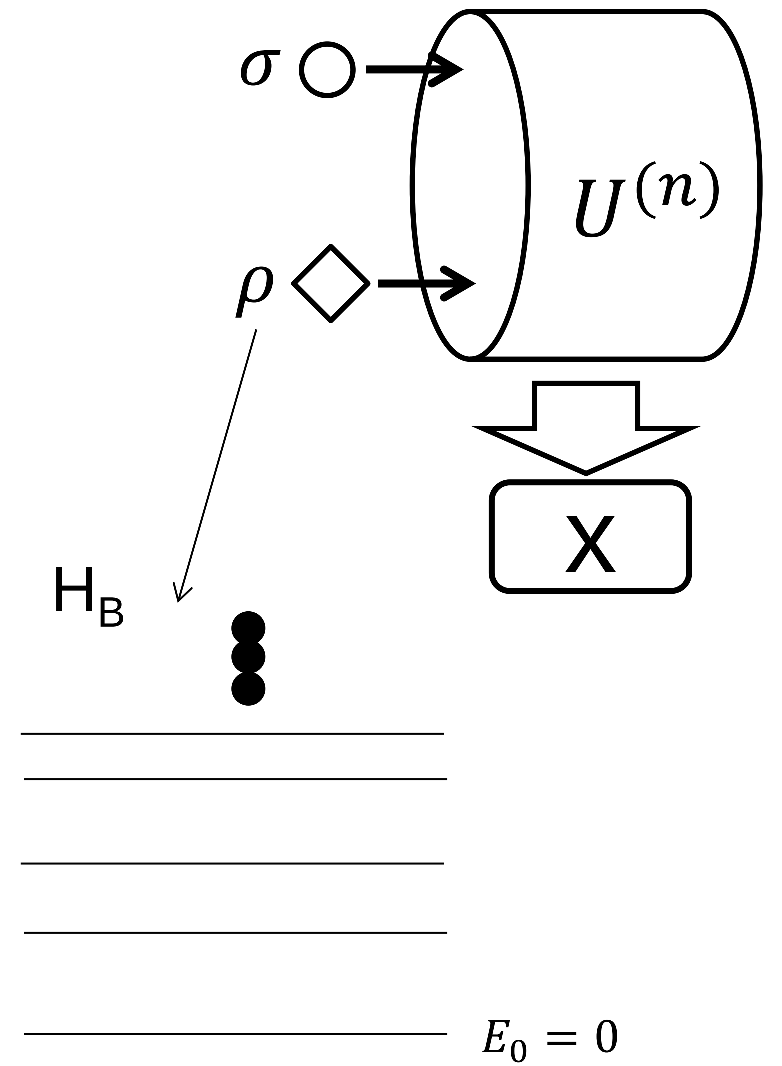




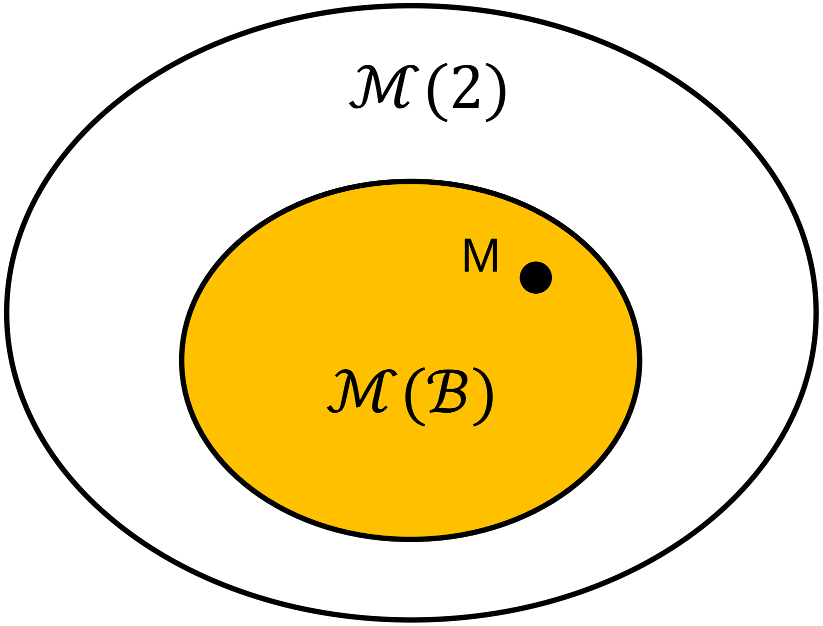
It is highly efficient: it allowed us to perform optimizations involving more than 4000 energy levels in a normal desktop.



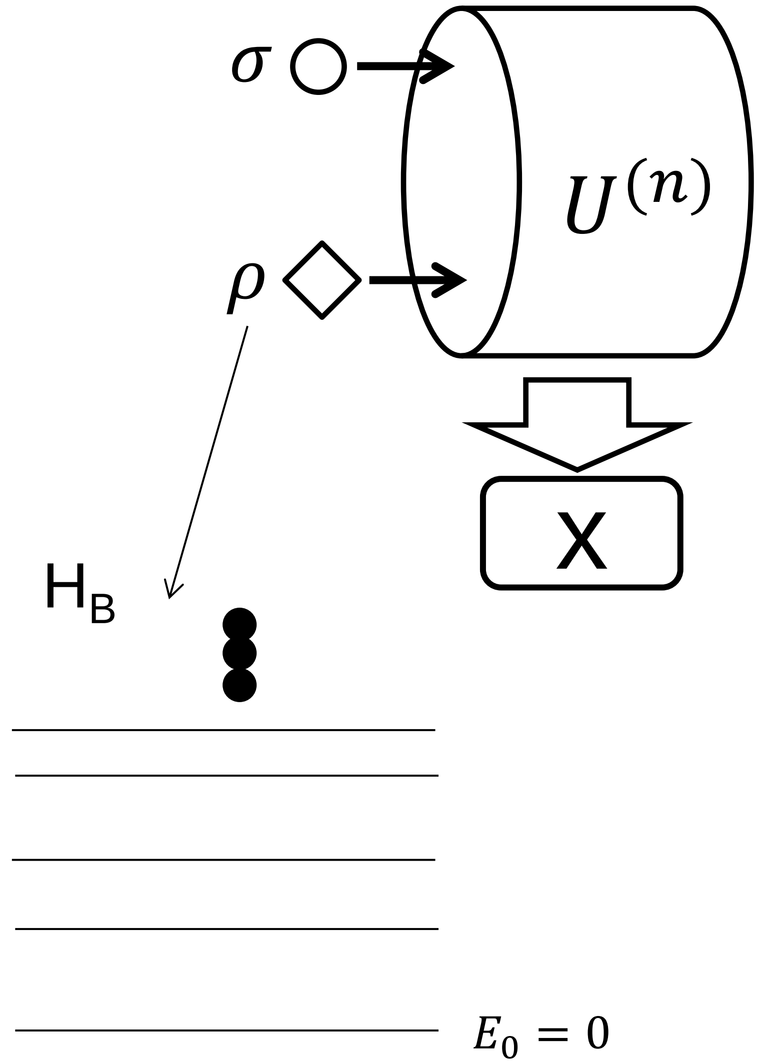




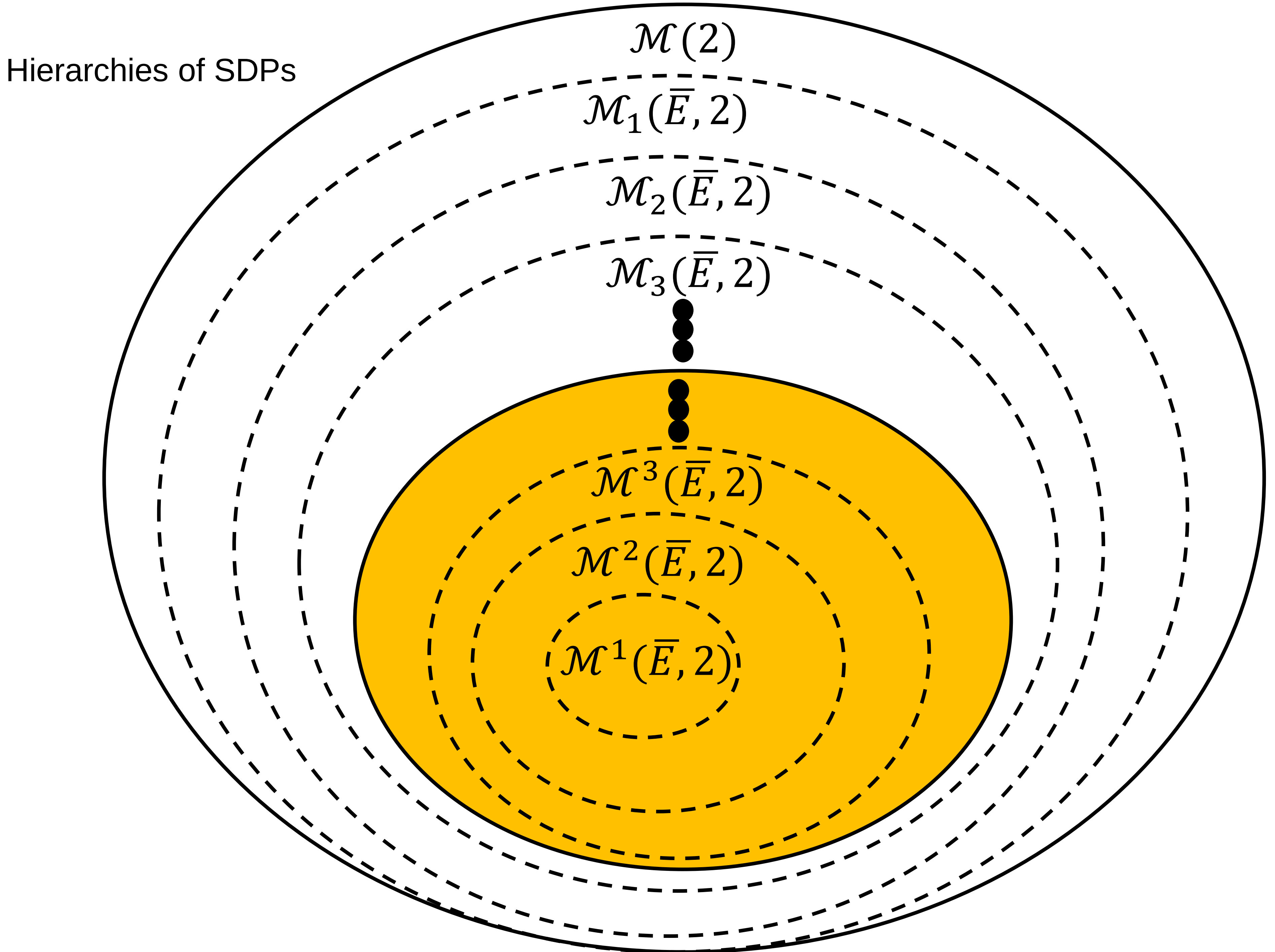
$$\int_0^\infty \varrho(E) E dE \le \bar{E}$$



Most likely, the membership problem cannot be decided by a single semidefinite program (SDP).



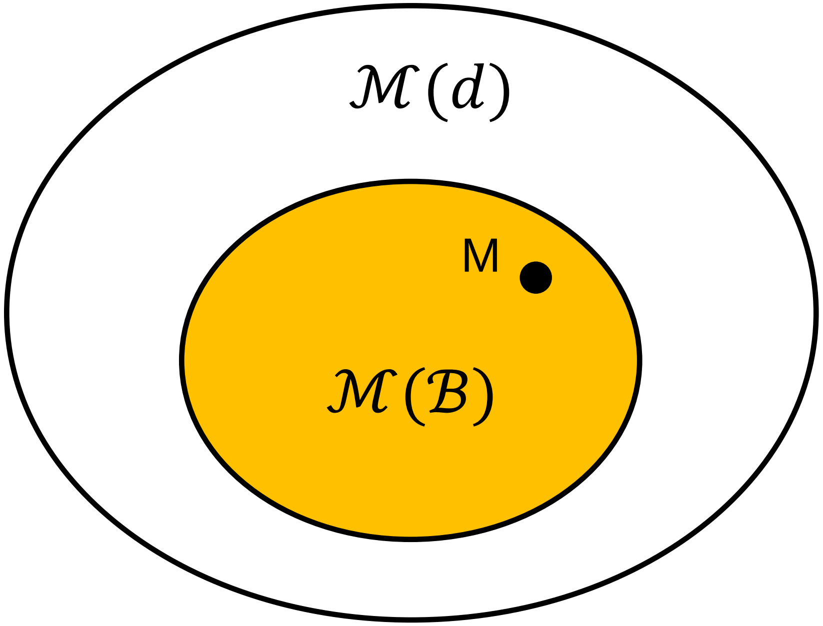
$$\int_0^\infty \varrho(E) E dE \le \bar{E}$$


Hierarchies of SDPs
$\mathcal{M}(2)$
$\mathcal{M}_1(\overline{E},2)$
$\mathcal{M}_2(\overline{E},2)$
$\mathcal{M}_3(\overline{E},2)$
$\mathcal{M}^3(\overline{E},2)$
$\mathcal{M}^2(\overline{E},2)$
$\mathcal{M}^1(\overline{E},2)$

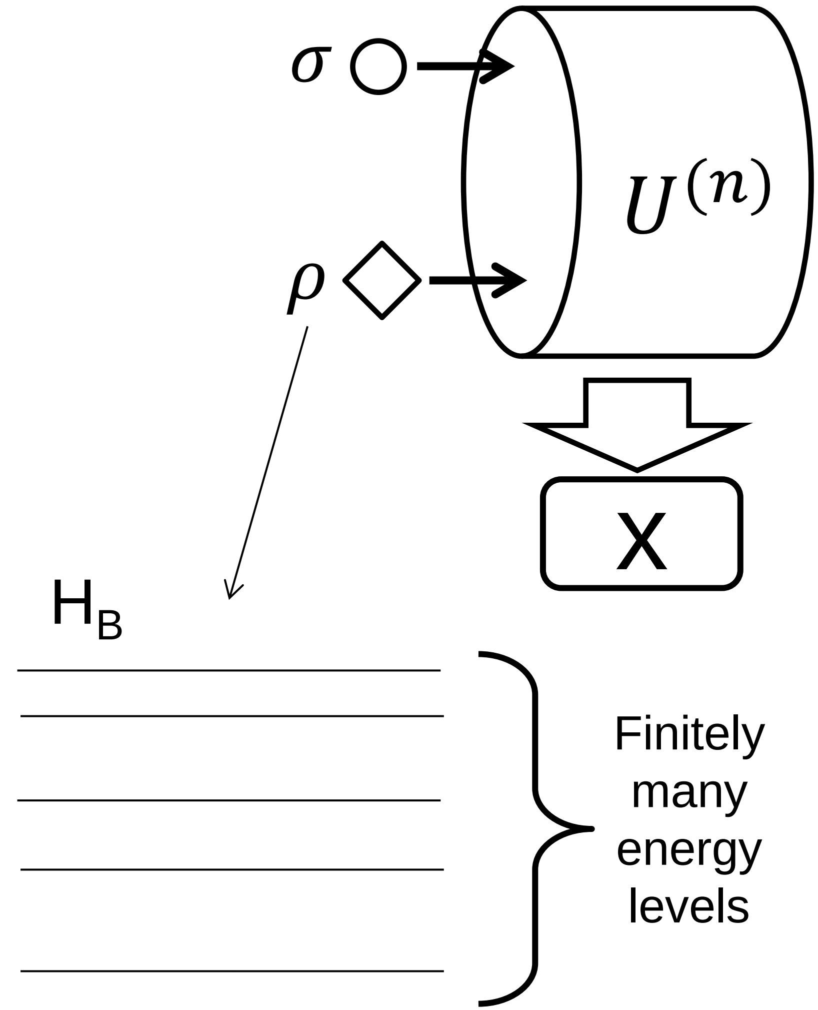
Hierarchies of SDPs

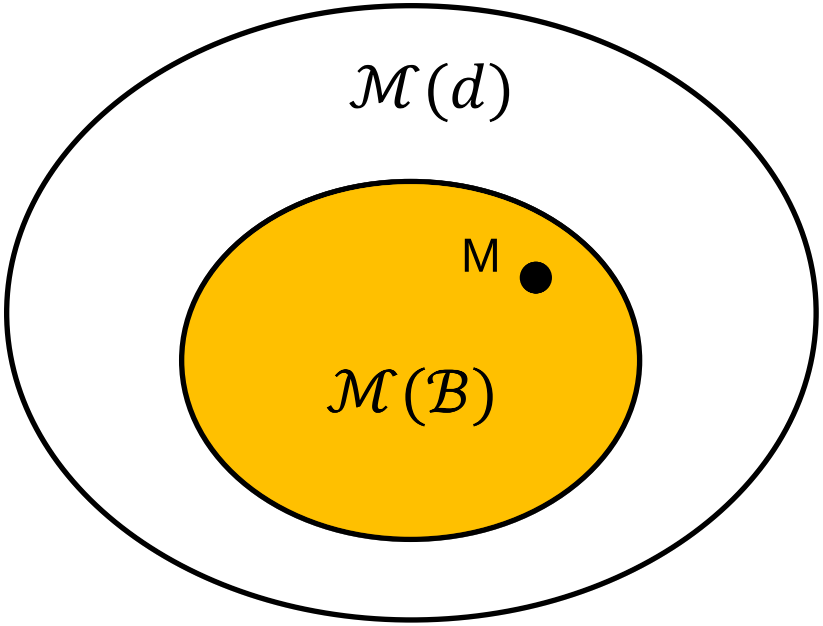
$$\varepsilon_Q[\mathcal{M}_d(\bar{E},2),\mathcal{M}^d(\bar{E},2)] \leq O\left(\frac{\Delta}{\bar{\bar{E}}d}\right)$$

Higher dimensions

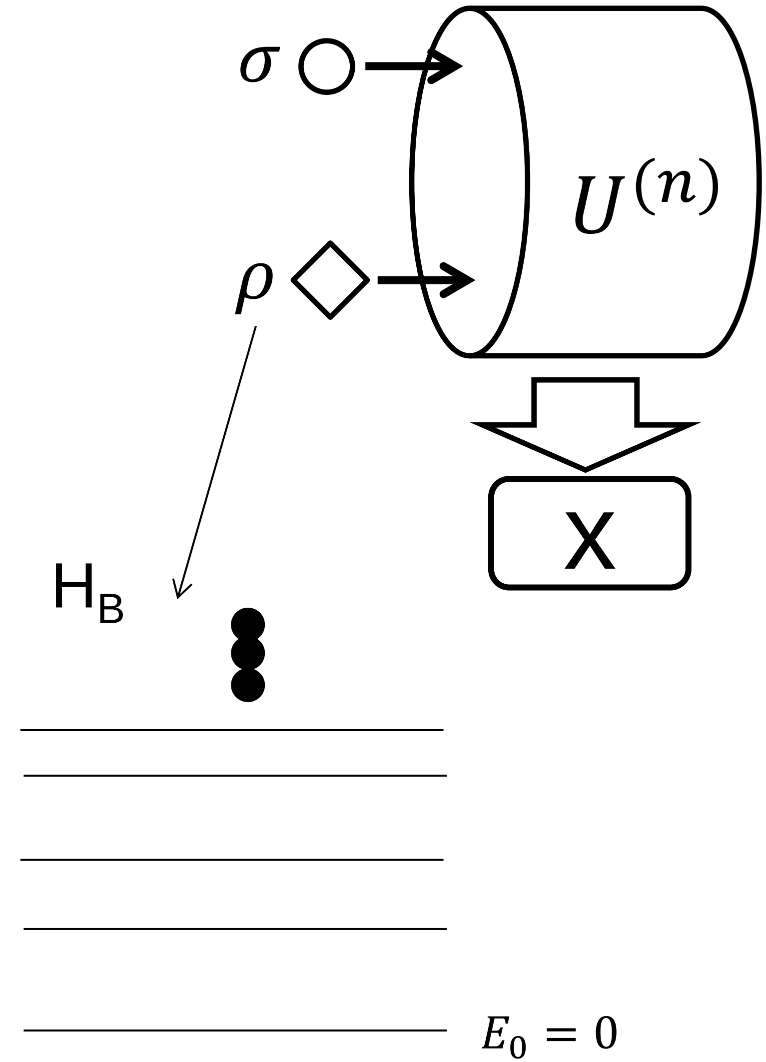


The membership problem can be decided by a single semidefinite program (SDP).





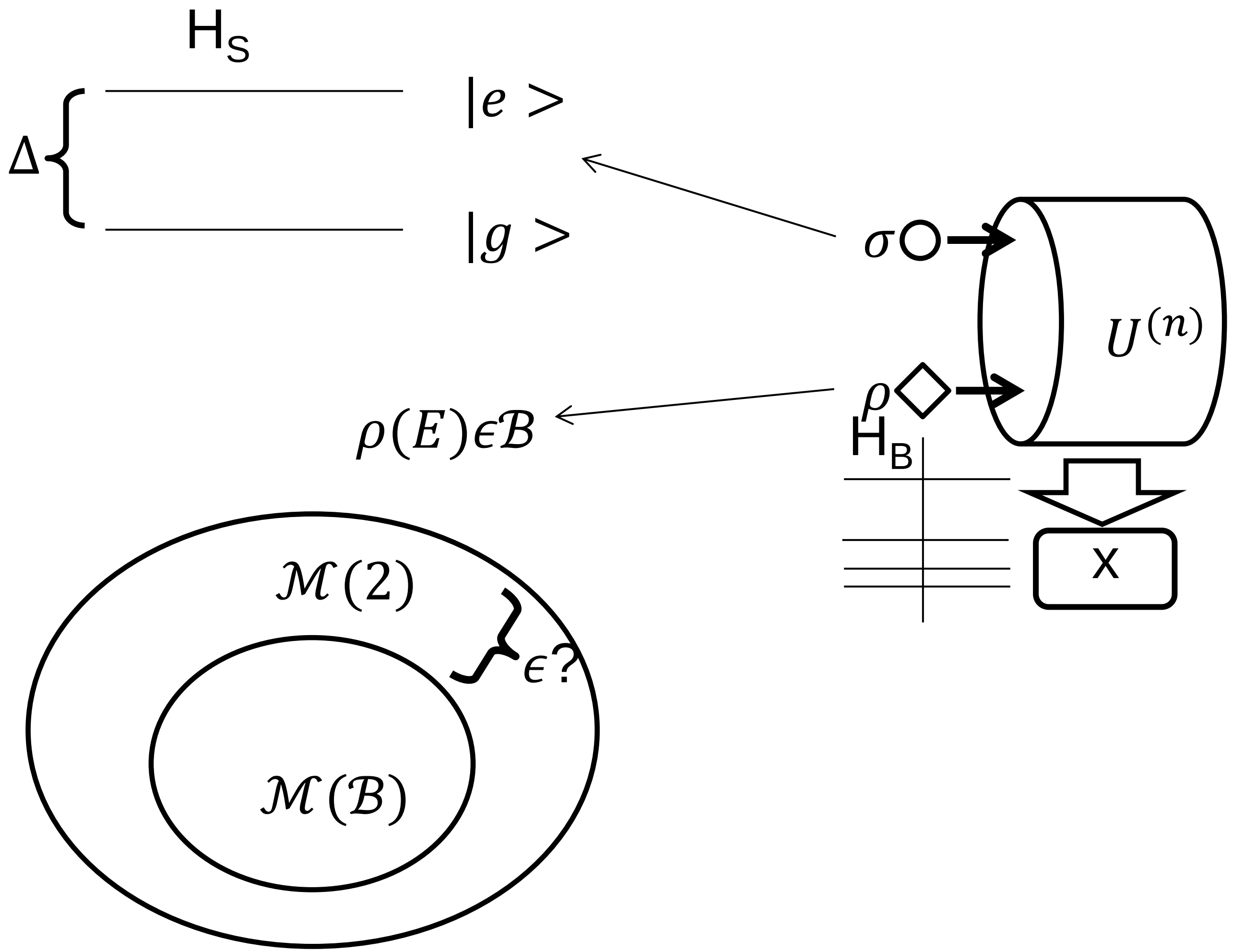
Hierarchy of SDPs



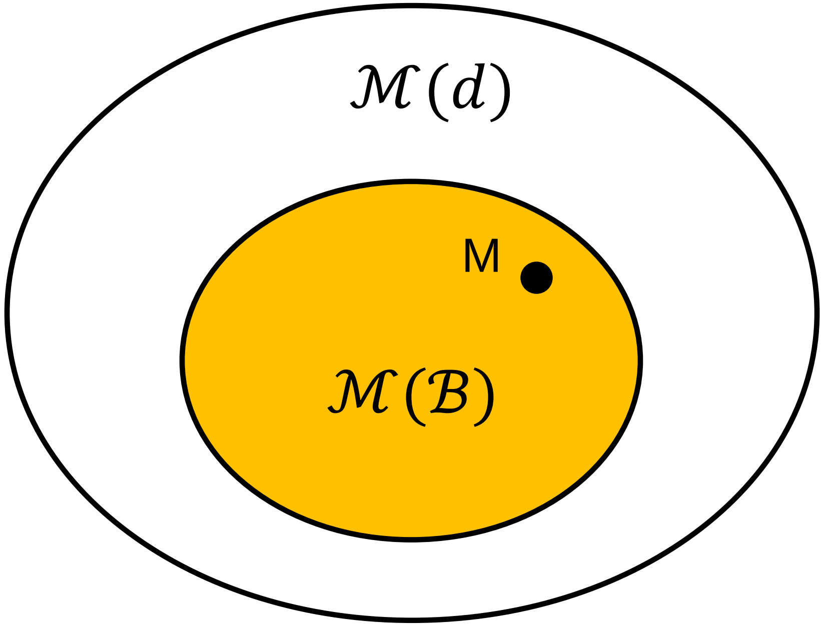
$$\int_0^\infty \varrho(E) E dE \leq \bar{E}$$

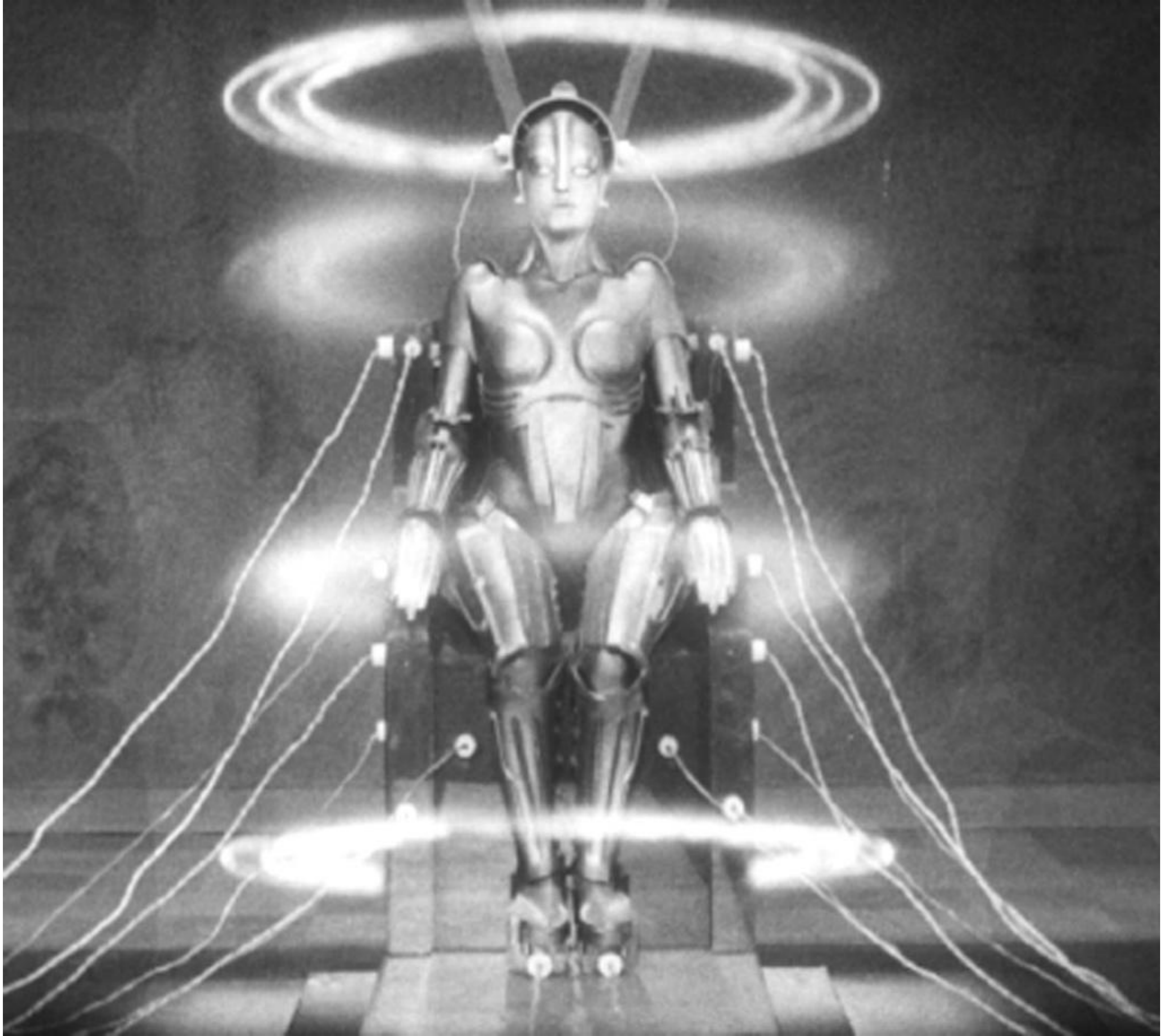
Conclusions

1) We have quantified how measurements of a qubit depend on the energy spectrum of the measurement device.

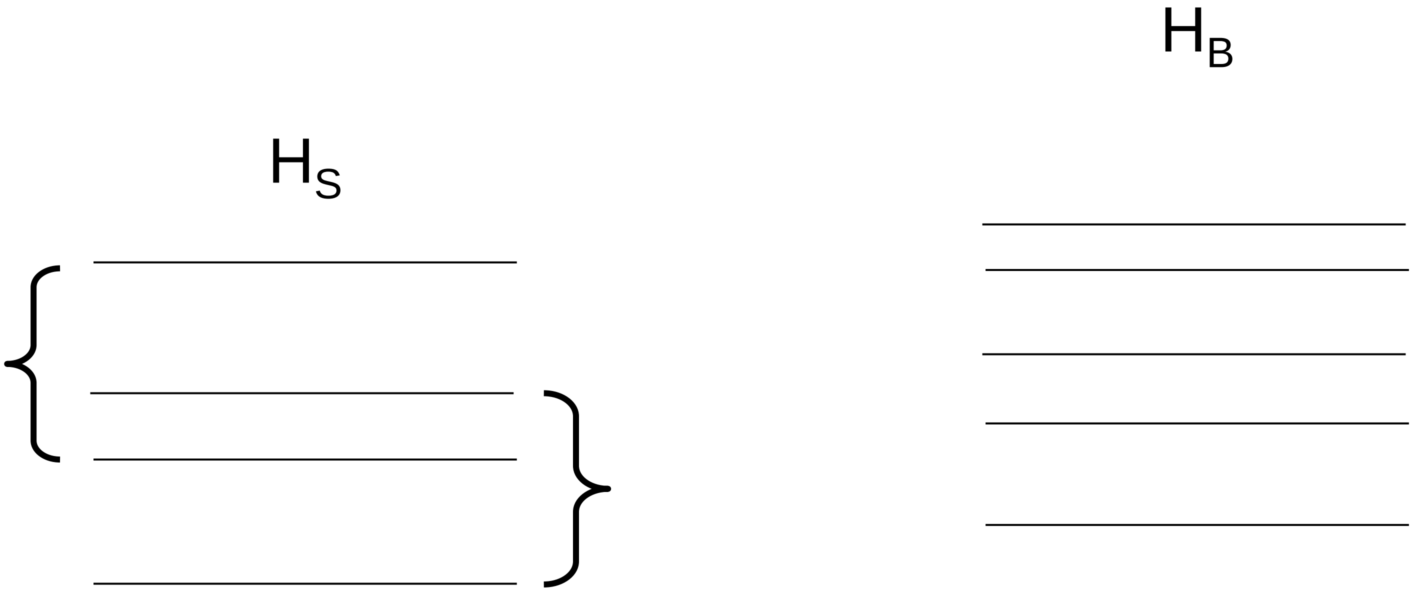


2) We have characterized measurements generated by measurement devices with reasonable assumptions on the energy spectrum, like finite energy or finite dimensionality.



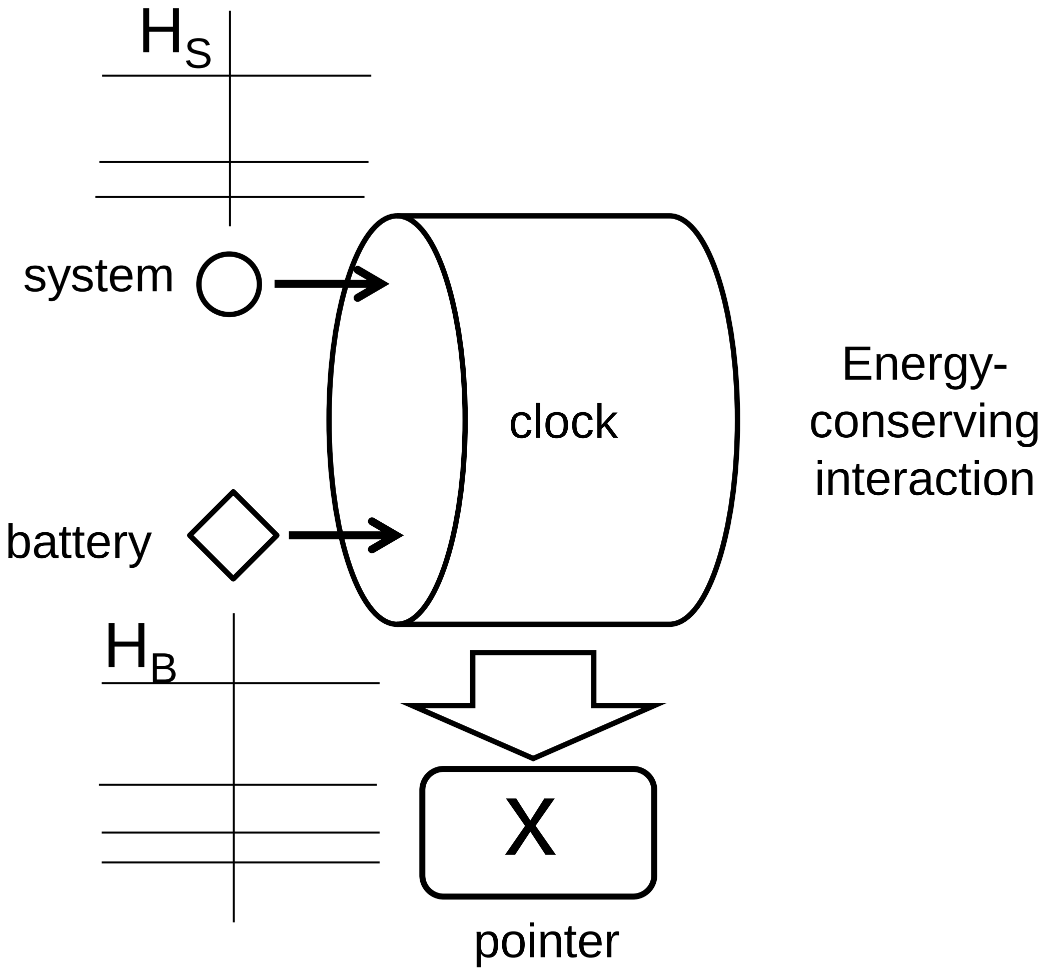


1) Study measurements in a qudit.



Effects of self-resonances?

2) Characterize thermodynamical operations



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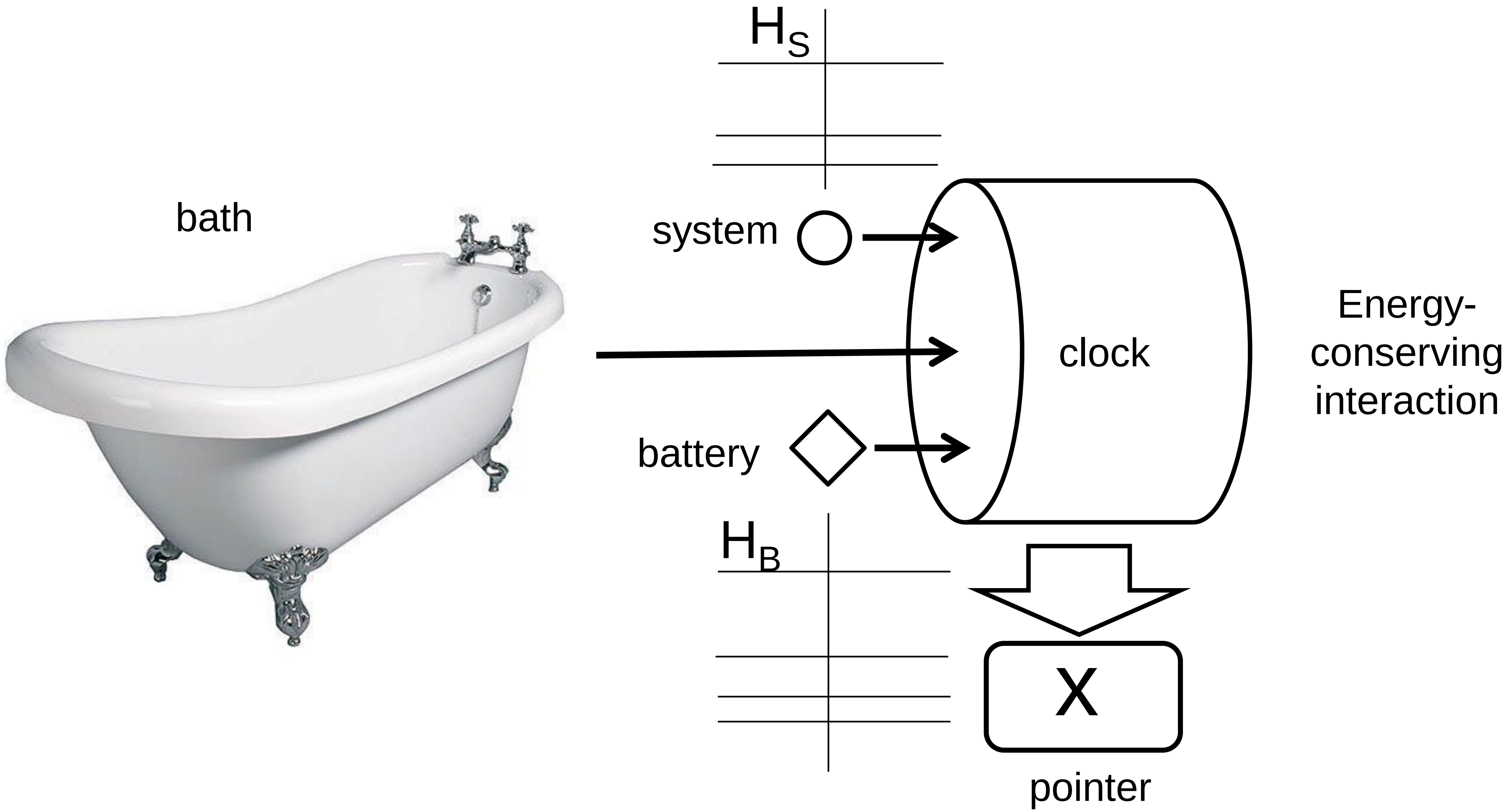

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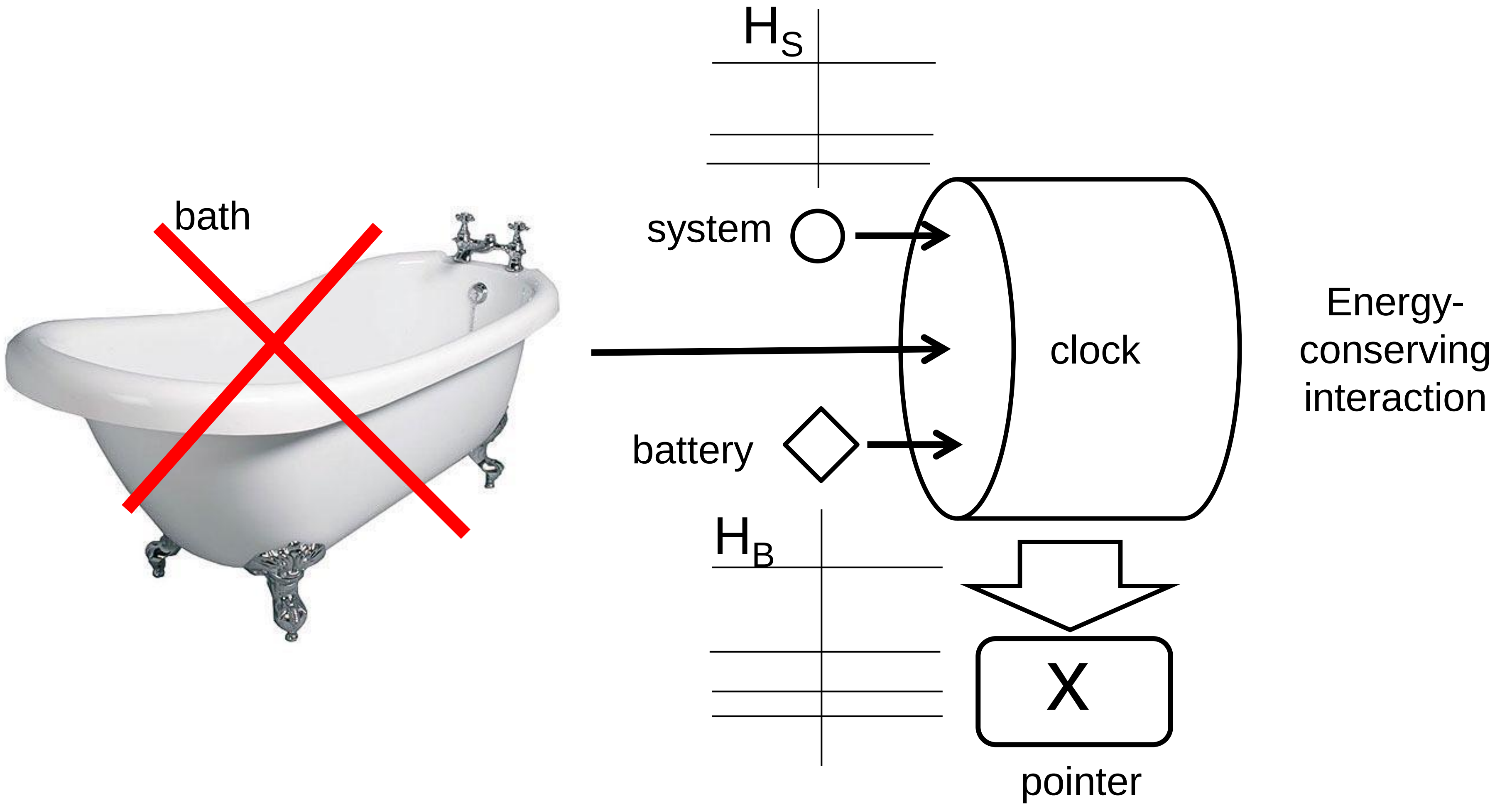

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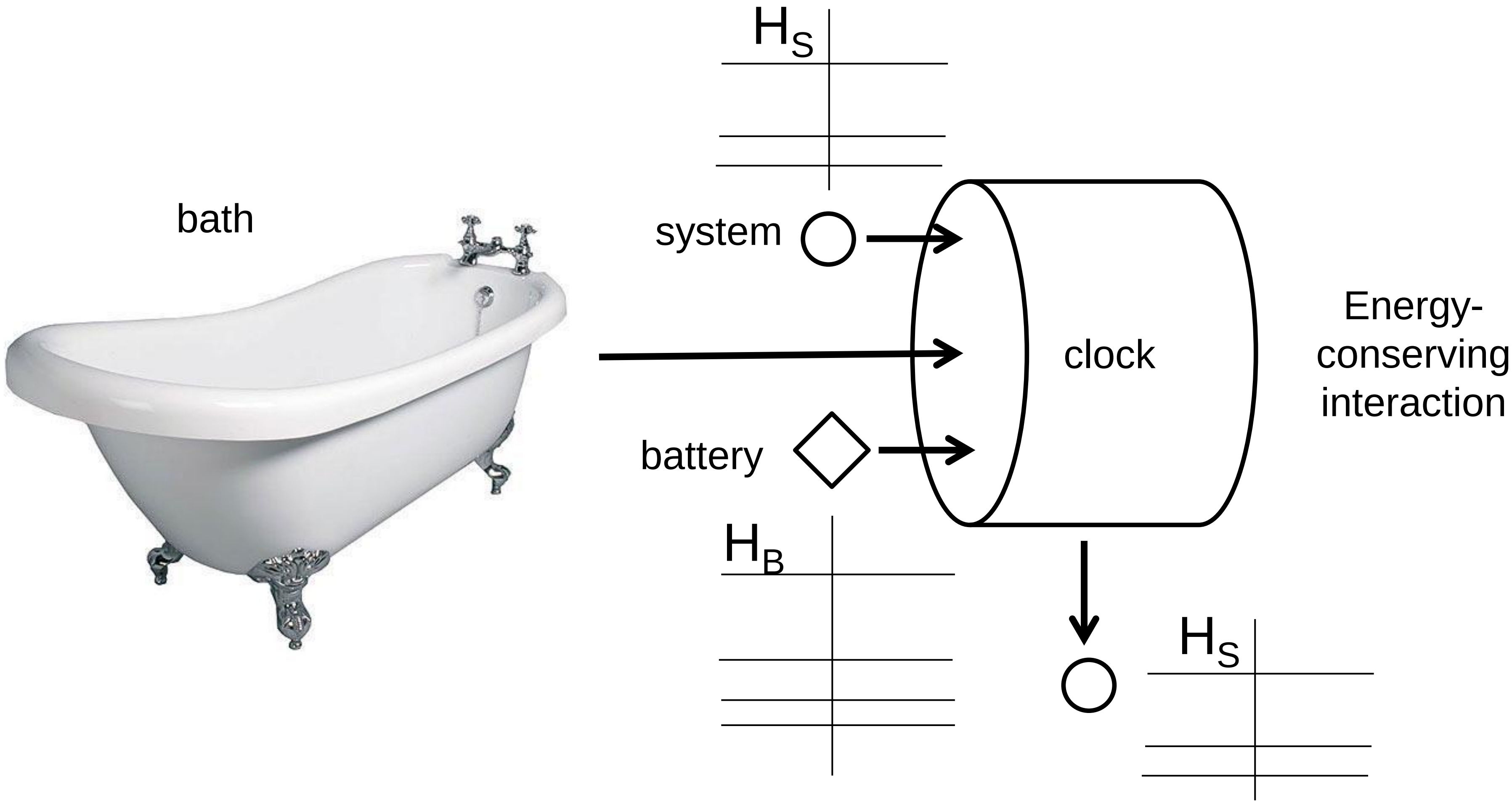

MEGAMAN HAS ENDED
THE EVIL DOMINATION
OF Dr.WILY
AND RESTORED
THE WORLD TO PEACE