Permutation invariant notions of multipartite entanglement and correlations

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Introduction

Bipartite correlation and entanglement

- classification/qualification/quantification: LO(CC)
- uncorrelated/correlated, and separable/entangled

Multipartite correlation and entanglement

- classification/qualification/quantification: LO(CC) too complicated
- "partial correlation/entanglement": finite, LO(CC)-compatible
- w.r.t. a splitting of the system (Level I.)
- w.r.t. possible splittings of the system (Level II.)
- disjoint classification of these (Level III.)

Permutation invariant properties

- three-level structure, Young-diagrams
- k-partitionability (k-separability), k-producibility (ent. depth), duality
- k-stretchability

Quantum states

States of discrete finite quantum systems

- *state vector:* $|\psi\rangle \in \mathcal{H}$ (normalized) superposition
- *pure state:* $\pi = |\psi\rangle\langle\psi| \in \mathcal{P}$
  we are uncertain about the outcomes of the measurement,
  pure states encode the *probabilities* of those
- inherent uncertainty
- *(mixed) state* (ensemble): $\rho = \sum_j p_j \pi_j \in \mathcal{D} = \mathrm{Conv}\,\mathcal{P}$, mixture
  we are uncertain about the pure state too
- $\mathcal{D}$ is convex, moreover, $\mathcal{P} = \mathrm{Extr}\,\mathcal{D}$
- "two-level probability theory"

Mixedness and distinguishability

Measure of mixedness

- von Neumann entropy: $S(\rho) = -\operatorname{Tr}\rho\ln\rho$
- concave, nonnegative, vanishes iff ρ pure
- Schur-concavity: *entropy* $=$ *mixedness*
- increasing in bistochastic quantum channels
- Schumacher's noiseless coding thm:

  von Neumann entropy $=$ *quantum information content*

Measure of distinguishability

- (Umegaki's) quantum relative entropy: $D(\rho||\sigma) = \operatorname{Tr}\rho(\ln\rho - \ln\sigma)$
- jointly convex, nonnegative, vanishes iff $\rho = \omega$
- quantum Stein's lemma: *relative entropy* $=$ *distinguishability*

  (rate of decaying of the probability of error in hypothesis testing, Hiai & Petz)
- decreasing in quantum channels

Bipartite correlation

Notions of correlation

- two events are correlated, if they occur more/less probably simultaneously than on their own: $p_{12} \neq p_1 p_2$
- measure of correlation of two prob.vars.:
  $\mathrm{COV}(A,B) = \langle (A - \langle A\rangle)(B - \langle B\rangle)\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle$
  $-1 \leq \mathrm{CORR}(A,B) = \mathrm{COV}(A,B)/\sqrt{\mathrm{VAR}(A)\,\mathrm{VAR}(B)} \leq 1$
- correlation "of the state itself": $\Gamma := \rho - \rho_1 \otimes \rho_2$
  then $\mathrm{COV}(A,B) = \mathrm{Tr}\,\Gamma A \otimes B = \langle \Gamma | A \otimes B\rangle_{\mathrm{HS}}$
- in q.m. there are many (nontrivially) different observables in a system
- Γ remains meaningful even if there are no values, only events
- the state is *uncorrelated* iff $\mathrm{COV}(A,B) = 0$ for all A, B,
  iff $\langle AB\rangle = \langle A\rangle\langle B\rangle$ for all A, B, iff $\rho = \rho_1 \otimes \rho_2$, iff $\Gamma = 0$

Bipartite correlation and entanglement

Pure states

- in the classical case, pure states are uncorrelated automatically, in the quantum case, they are not!
- if a pure state is correlated, then this correlation is of quantum origin, and this is what we call *entanglement*
- $|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \quad \rightsquigarrow \quad |\psi\rangle\langle\psi| = \pi \in \mathcal{P}$
- uncorrelated: *separable*
  $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \quad \rightsquigarrow \quad \pi = \pi_1 \otimes \pi_2 \in \mathcal{P}_{\text{sep}} \subset \mathcal{P}$
- correlated: *entangled* $(\mathcal{P} \setminus \mathcal{P}_{\text{sep}})$
  Then measurement on a subsystem "causes"$^?$ the collapse of the state of the other. (worry of EPR)
- state of subsystem (e.g., $\operatorname{Tr}_2 \pi \in \mathcal{D}_1$) not necessarily pure
- π is entangled if (and only if) $\operatorname{Tr}_2 \pi$ and $\operatorname{Tr}_1 \pi$ are mixed
  In this case, *"the best possible knowledge of the whole does not involve the best possible knowledge of its parts."* (Schrödinger)

Bipartite correlation and entanglement

Mixed states: correlation

- *uncorrelated:* $\Gamma = 0$ (product), $\rho = \rho_1 \otimes \rho_2 \in \mathcal{D}_{\mathrm{unc}}$,
  else *correlated* $(\mathcal{D} \setminus \mathcal{D}_{\mathrm{unc}})$
- easy to decide

Mixed states: entanglement

- *separable:* there exists separable decomposition:
$$\rho = \sum_i p_i \pi_{1,i} \otimes \pi_{2,i} \in \mathcal{D}_{\mathrm{sep}} = \mathrm{Conv}\, \mathcal{P}_{\mathrm{sep}} = \mathrm{Conv}\, \mathcal{D}_{\mathrm{unc}} \subset \mathcal{D}$$
- classically correlated sources produce states of this kind (Werner)
  preparable by Local Operations and Classical Communication (LOCC),
  else *entangled* $(\mathcal{D} \setminus \mathcal{D}_{\mathrm{sep}})$
- the decomposition is not unique
- deciding separability is difficult

Bipartite correlation and entanglement – measures

- correlation "of the state itself": $\Gamma := \rho - \rho_1 \otimes \rho_2$
  then $\mathrm{COV}(\rho; A, B) = \langle AB \rangle - \langle A \rangle \langle B \rangle = \mathrm{Tr}\, \Gamma A \otimes B = \langle \Gamma | A \otimes B \rangle_{\mathrm{HS}}$
- uncorrelated: $\Gamma = 0$
- correlation measures, based on geometry:
  by *distance* (metric from norm): $C_q(\rho) = \|\Gamma\|_q = D_q(\rho, \rho_1 \otimes \rho_2)$
  or by *distinguishability* (rel. entr.): $C(\rho) = D(\rho \| \rho_1 \otimes \rho_2) =$
  leads to the *mutual information* $= S(\rho_1) + S(\rho_2) - S(\rho) = I_{1|2}(\rho)$
- for the latter one, we have another, stronger motivation:

$$\min_{\sigma \in \mathcal{D}_{\mathrm{unc}}} D(\rho \| \sigma) = D(\rho \| \rho_1 \otimes \rho_2)$$

"how correlated = how not uncorrelated = how distinguishable from the uncorrelated ones"

- correlation might not be seen well from COV, but for all A, B,

$$\frac{1}{2} \mathrm{COV}(\rho; \hat{A}, \hat{B})^2 \leq C(\rho), \qquad \hat{A} = A / \|A\|_\infty, \hat{B} = B / \|B\|_\infty$$

Bipartite correlation and entanglement – measures

- *correlation* (mutual information):

$$C(\rho) = \min_{\sigma \in \mathcal{D}_{\text{unc}}} D(\rho||\sigma) = S(\rho_1) + S(\rho_2) - S(\rho)$$

"how correlated = how not uncorrelated"

- *entanglement* (for pure states) *entanglement of formation* (for mixed states):

$$E(\pi) = C|_{\mathcal{P}}(\pi), \qquad E(\rho) = \min\Big\{\sum_i p_i E(\pi_i) \,\Big|\, \sum_i p_i \pi_i = \rho\Big\}$$

for pure states: entanglement = correlation

$E(\pi) = 2S(\pi_1) = 2S(\pi_2)$, "2×entanglement entropy"

for mixed states: average entanglement of the optimal decomposition

LOCC-monotone (proper entanglement measure)

- faithful: $C(\rho) = 0 \Leftrightarrow \rho \in \mathcal{D}_{\text{unc}}$, $E(\rho) = 0 \Leftrightarrow \rho \in \mathcal{D}_{\text{sep}}$
- $E(\rho)$ is hard to calculate

Multipartite correlation and entanglement – structure

Level 0.: subsystems — Boolean lattice structure: $P_0 = 2^L$

- whole system: $L = \{1, 2, \ldots, n\}$
- subsystem: $X \subseteq L$, then $\mathcal{H}_X$, $\mathcal{P}_X$, $\mathcal{D}_X$

Level I.: partitions — lattice structure: $P_{\mathrm{I}} = \Pi(L)$

- partition: $\xi = \{X_1, X_2, \ldots, X_{|\xi|}\} \in \Pi(L)$
- refinement (partial order): $\upsilon \preceq \xi$ def.: $\forall Y \in \upsilon, \exists X \in \xi : Y \subseteq X$
- *ξ-uncorrelated states:* $\mathcal{D}_{\xi\text{-unc}} = \left\{ \bigotimes_{X \in \xi} \rho_X \right\}$
  LO-closed
  $\upsilon \preceq \xi \Leftrightarrow \mathcal{D}_{\upsilon\text{-unc}} \subseteq \mathcal{D}_{\xi\text{-unc}}$
- *ξ-separable states:* $\mathcal{D}_{\xi\text{-sep}} = \mathrm{Conv}\, \mathcal{D}_{\xi\text{-unc}}$
  LOCC-closed
  $\upsilon \preceq \xi \Leftrightarrow \mathcal{D}_{\upsilon\text{-sep}} \subseteq \mathcal{D}_{\xi\text{-sep}}$

Szalay, Barcza, Szilvási, Veis, Legeza, SciRep **7**, 2237 (2017)
Szalay, PRA **92**, 042329 (2015)
Szalay, Kökényesi, PRA **86**, 032341 (2012)
Seevinck, Uffink, PRA **78**, 032101 (2008)
Dür, Cirac, Tarrach, PRL **83**, 3562 (1999)

Multipartite correlation and entanglement – structure

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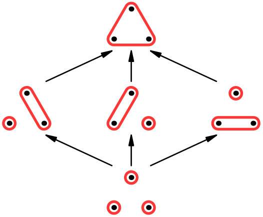
  $n = 3$:



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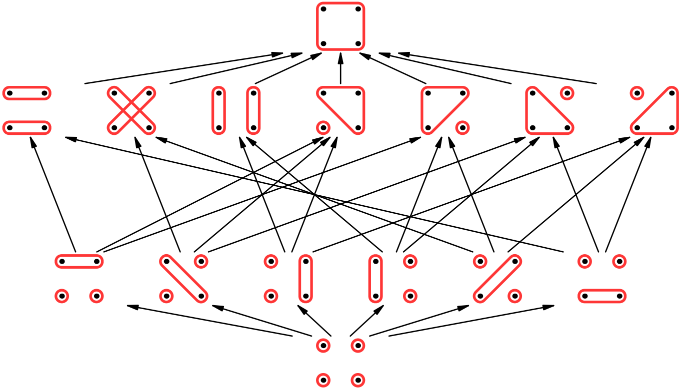

Multipartite correlation and entanglement – measures

Level I.: partitions lattice structure: $P_{\text{I}} = \Pi(L)$

- *ξ-correlation* (ξ-mutual information):

$$C_\xi(\rho) = \min_{\sigma \in \mathcal{D}_{\xi\text{-unc}}} D(\rho||\sigma) = \sum\nolimits_{X \in \xi} S(\rho_X) - S(\rho)$$

LO-monotone (proper correlation measure)

- *ξ-entanglement (of formation):*

$$E_\xi(\pi) = C_\xi|_{\mathcal{P}}(\pi), \qquad E_\xi(\rho) = \min\Big\{\sum\nolimits_i p_i E_\xi(\pi_i) \,\Big|\, \sum\nolimits_i p_i \pi_i = \rho\Big\}$$

LOCC-monotone (proper entanglement measure)

- faithful: $C_\xi(\rho) = 0 \Leftrightarrow \rho \in \mathcal{D}_{\xi\text{-unc}}$, $E_\xi(\rho) = 0 \Leftrightarrow \rho \in \mathcal{D}_{\xi\text{-sep}}$
- multipartite monotone: $\upsilon \preceq \xi \Leftrightarrow C_\upsilon \geq C_\xi, E_\upsilon \geq E_\xi$

Szalay, Barcza, Szilvási, Veis, Legeza, SciRep **7**, 2237 (2017)
Szalay, PRA **92**, 042329 (2015)

Multipartite correlation and entanglement – structure

Level II.: multiple partitions lattice structure: $P_{\mathrm{II}} = \mathcal{O}_{\downarrow}(P_{\mathrm{I}}) \setminus \{\emptyset\}$

- partition ideal: $\boldsymbol{\xi} = \{\xi_1, \xi_2, \dots, \xi_{|\boldsymbol{\xi}|}\} \subseteq P_{\mathrm{I}}$, closed downwards w.r.t. $\preceq$
- partial order: $\boldsymbol{\upsilon} \preceq \boldsymbol{\xi}$ def.: $\boldsymbol{\upsilon} \subseteq \boldsymbol{\xi}$
- *$\boldsymbol{\xi}$-uncorrelated states:* $\mathcal{D}_{\boldsymbol{\xi}\text{-unc}} = \cup_{\xi\in\boldsymbol{\xi}} \mathcal{D}_{\xi\text{-unc}}$
  LO-closed $\boldsymbol{\upsilon} \preceq \boldsymbol{\xi} \Leftrightarrow \mathcal{D}_{\boldsymbol{\upsilon}\text{-unc}} \subseteq \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$
- *$\boldsymbol{\xi}$-separable states:* $\mathcal{D}_{\boldsymbol{\xi}\text{-sep}} = \operatorname{Conv} \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$
  LOCC-closed $\boldsymbol{\upsilon} \preceq \boldsymbol{\xi} \Leftrightarrow \mathcal{D}_{\boldsymbol{\upsilon}\text{-sep}} \subseteq \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$
- spec.: k-partitionable and k'-producible (chains)

$$\boldsymbol{\mu}_k = \big\{\mu \in P_{\mathrm{I}} \;\big|\; |\mu| \geq k\big\}, \qquad \boldsymbol{\nu}_{k'} = \big\{\nu \in P_{\mathrm{I}} \;\big|\; \forall N \in \nu : |N| \leq k'\big\}$$

- with these:
  k-partitionably and *k'-producibly uncorrelated*
  k-partitionably and *k'-producibly separable* states

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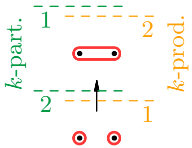

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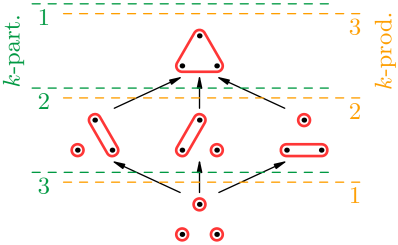

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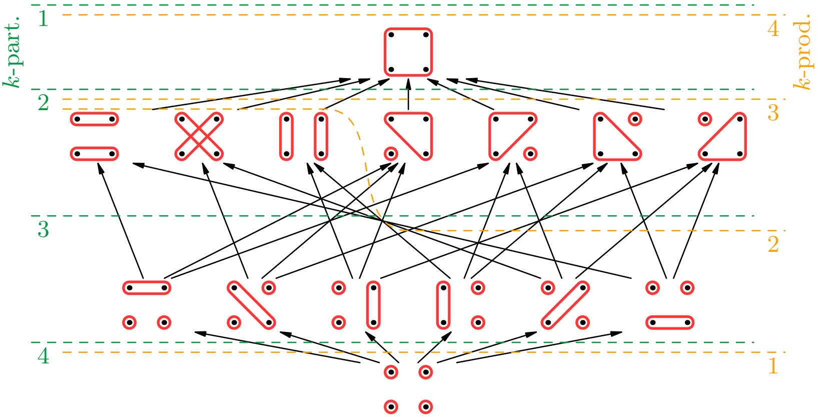

Multipartite correlation and entanglement – measures

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- *ξ-correlation*:

$$C_{\boldsymbol{\xi}}(\rho) = \min_{\sigma\in\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}} D(\rho||\sigma) = \min_{\xi\in\boldsymbol{\xi}} C_\xi(\rho)$$

  LO-monotone (proper correlation measure)

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- faithful: $C_{\boldsymbol{\xi}}(\rho) = 0 \Leftrightarrow \rho \in \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$, $E_{\boldsymbol{\xi}}(\rho) = 0 \Leftrightarrow \rho \in \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$
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- spec.: k-particionability and k'-producibility
  k-partitionability and *k'-producibility correlation*
  k-partitionability and *k'-producibility entanglement*

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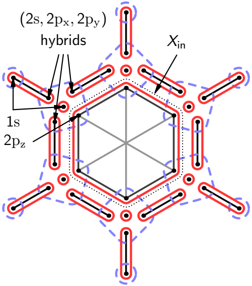
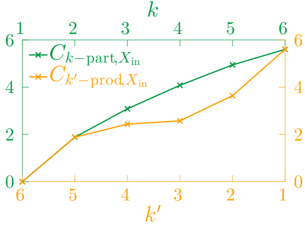
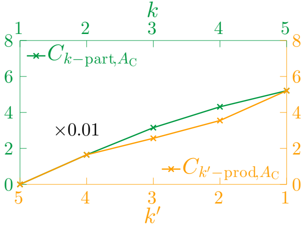
Example: Electron system of molecules

- elementary subsystems: localized atomic orbitals (Pipek-Mezey)
- “atomic split”: $\alpha = \{A_1, A_2, \dots, A_{|\alpha|}\}$ (blue)
- “bond split”: $\beta = \{B_1, B_2, \dots, B_{|\beta|}\}$ (red)

benzene (C_6H_6): $C_\alpha = 29.52$, $C_\beta = 2.33$







(in units ln 4)

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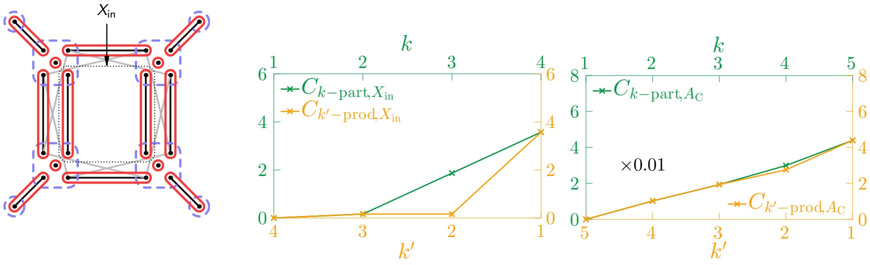
- "bond split": $\beta = \{B_1, B_2, \ldots, B_{|\beta|}\}$ (red)

cyclobutadiene (C_4H_4): $C_\alpha = 19.48$, $C_\beta = 3.17$



(in units ln 4)

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Entanglement classes

Level III: Entanglement classes lattice structure: $P_{\mathrm{III}} = \mathcal{O}_{\uparrow}(P_{\mathrm{II}}) \setminus \{\emptyset\}$

- ideal filter: $\underline{\boldsymbol{\xi}} = \{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_{|\underline{\boldsymbol{\xi}}|}\} \subseteq P_{\mathrm{II}}$ (closed upwards w.r.t. $\preceq$)
- partial order: $\underline{\boldsymbol{\upsilon}} \preceq \underline{\boldsymbol{\xi}}$ def.: $\underline{\boldsymbol{\upsilon}} \subseteq \underline{\boldsymbol{\xi}}$
- partial separability classes: intersections of $\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$

$$\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-sep}} := \bigcap_{\boldsymbol{\xi} \notin \underline{\boldsymbol{\xi}}} \overline{\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}} \cap \bigcap_{\boldsymbol{\xi} \in \underline{\boldsymbol{\xi}}} \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$$

- LOCC convertibility:
  if $\exists \rho \in \mathcal{C}_{\underline{\boldsymbol{\upsilon}}}$, $\exists \Lambda$ LOCC map s.t. $\Lambda(\rho) \in \mathcal{C}_{\underline{\boldsymbol{\xi}}}$ then $\underline{\boldsymbol{\upsilon}} \preceq \underline{\boldsymbol{\xi}}$

Szalay, PRA **92**, 042329 (2015)

Entanglement classes

Level III: Entanglemen

- ideal filter: $\underline{\boldsymbol{\xi}} = \{$
- partial order: $\underline{\boldsymbol{\upsilon}} \preceq$
- partial separability

- LOCC convertibili
  if $\exists \rho \in \mathcal{C}_{\underline{\boldsymbol{\upsilon}}}$, $\exists \Lambda$ LO

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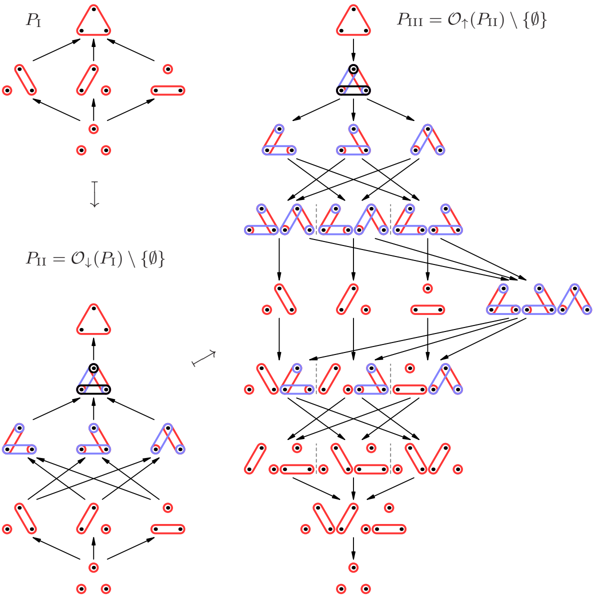

Correlation classes

Level III: Corr./Ent. classes   lattice structure: $P_{\mathrm{III}} = \mathcal{O}_{\uparrow}(P_{\mathrm{II}}) \setminus \{\emptyset\}$

- partial correlation classes: intersections of $\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$

$$\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-unc}} := \bigcap_{\boldsymbol{\xi}\notin\underline{\boldsymbol{\xi}}} \overline{\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}} \cap \bigcap_{\boldsymbol{\xi}\in\underline{\boldsymbol{\xi}}} \mathcal{D}_{\boldsymbol{\xi}\text{-unc}} \neq \emptyset$$

iff $\underline{\boldsymbol{\xi}} = \uparrow\{\downarrow\{\xi\}\}$ (proven)

Szalay, JPhysA **51**, 485302 (2018)

- partial separability classes: intersections of $\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$

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for all $\underline{\boldsymbol{\xi}}$ (conjectured)

  proven constructively for $n = 3$ Han, Kye, PRA **99**, 032304 (2019)

- LO convertibility:
  if $\exists\rho \in \mathcal{C}_{\underline{\boldsymbol{\upsilon}}\text{-unc}}$, $\exists\Lambda$ LO map s.t. $\Lambda(\rho) \in \mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-unc}}$ then $\underline{\boldsymbol{\upsilon}} \preceq \underline{\boldsymbol{\xi}}$
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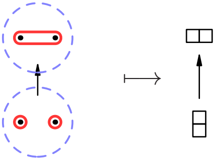
Level I.: splitting type of the system of n elementary subsystems

- integer partition $\hat{\xi} = \{x_1, x_2, \dots, x_{|\hat{\xi}|}\}$ of n (multiset) *(Young diag.)*
- coarser/finer: $\sqsubseteq$ partial order: $\hat{\upsilon} \sqsubseteq \hat{\xi}$ if exist $\upsilon \preceq \xi$ of those types
- this is a new partial order, $\top$, $\bot$, not a lattice $\hat{P}_{\mathrm{I}}$

$n = 2$:



Szalay, Quantum **3**, 204 (2019)

Permutation symmetric correlation and entanglement

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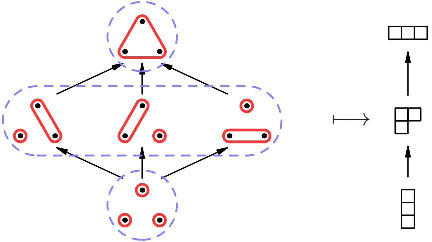

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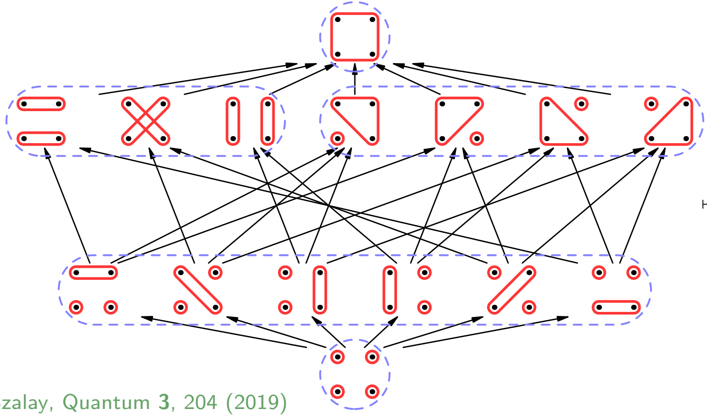
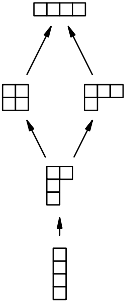
$n = 4$:



$\longmapsto$



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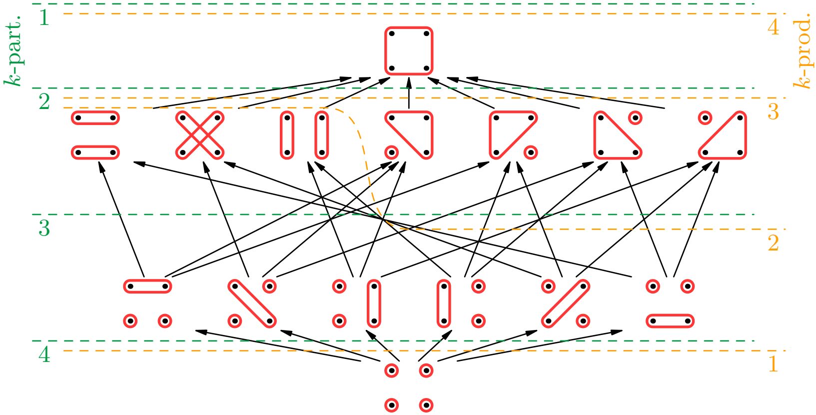
Permutation symmetric correlation and entanglement

Structure of k-partitionability and k'-producibility

- P_{I} graded lattice, gradation = partitionability
- what is producibility? a kind of dual property: natural conjugation



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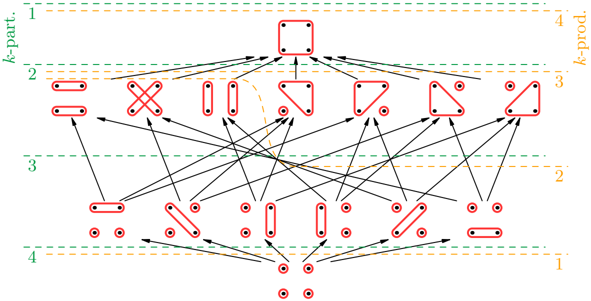
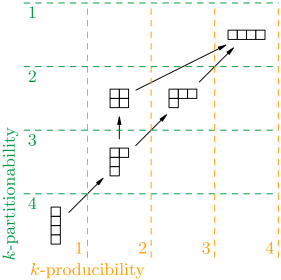
Permutation symmetric correlation and entanglement

Structure of k-partitionability and k'-producibility

- P_{I} graded lattice, gradation = partitionability
- what is producibility? a kind of dual property: natural conjugation





- part. / prod. = minimal height / maximal width of Young diagram
- note: $\sqsubseteq$ is not respected by the conjugation

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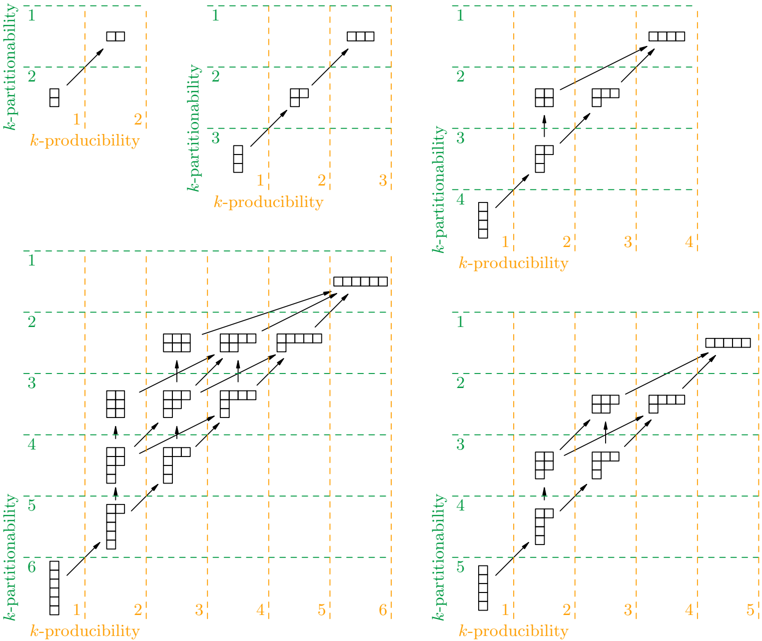

k-partitionability
k-producibility
1
2
3
4
5
6

Construction

- perm. symmetric *properties*, not only for perm. symmetric *states*
- $s(X) := |X|$, and elementwisely on P_{I}, works also for P_{II} and P_{III}
- the construction is well-defined — Szalay, Quantum **3**, 204 (2019)

$$\begin{array}{ccc}
(P_{\mathrm{III}},\preceq) & \xrightarrow{\;s\;} & (\hat{P}_{\mathrm{III}},\sqsubseteq) \\
\uparrow \mathcal{O}_{\uparrow}\setminus\{\emptyset\} & & \uparrow \mathcal{O}_{\uparrow}\setminus\{\emptyset\} \\
(P_{\mathrm{II}},\preceq) & \xrightarrow{\;s\;} & (\hat{P}_{\mathrm{II}},\sqsubseteq) \\
\uparrow \mathcal{O}_{\downarrow}\setminus\{\emptyset\} & & \uparrow \mathcal{O}_{\downarrow}\setminus\{\emptyset\} \\
(P_{\mathrm{I}},\preceq) & \xrightarrow{\;s\;} & (\hat{P}_{\mathrm{I}},\sqsubseteq)
\end{array}$$

- state sets $\mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-unc}}$, $\mathcal{D}_{\hat{\boldsymbol{\xi}}\text{-sep}}$, — inclusion hierarchy works well
- measures $C_{\hat{\boldsymbol{\xi}}}(\rho)$, $E_{\hat{\boldsymbol{\xi}}}(\rho)$, — multipartite monotonicity works well
- classes $\mathcal{C}_{\underline{\hat{\boldsymbol{\xi}}}\text{-unc}}$, $\mathcal{C}_{\underline{\hat{\boldsymbol{\xi}}}\text{-sep}}$, — LO(CC) convertibility works well

height, width and rank of a Young diagram $\Longrightarrow$ properties

$$h(\hat{\xi}) := |\hat{\xi}| \qquad \hat{\boldsymbol{\mu}}_k = \{\hat{\mu} \in \hat{P}_{\mathrm{I}} \mid h(\hat{\mu}) \geq k\}$$

$$w(\hat{\xi}) := \max \hat{\xi} \qquad \hat{\boldsymbol{\nu}}_k = \{\hat{\nu} \in \hat{P}_{\mathrm{I}} \mid w(\hat{\nu}) \leq k\}$$

$$r(\hat{\xi}) := w(\hat{\xi}) - h(\hat{\xi}) \qquad \hat{\boldsymbol{\tau}}_k = \{\hat{\tau} \in \hat{P}_{\mathrm{I}} \mid r(\hat{\tau}) \leq k\}$$

- monotones

$$\hat{\upsilon} \prec \hat{\xi} \quad \Longrightarrow \quad h(\hat{\upsilon}) > h(\hat{\xi}), \quad w(\hat{\upsilon}) \leq w(\hat{\xi}), \quad r(\hat{\upsilon}) < r(\hat{\xi}).$$

- chains

$$\hat{\boldsymbol{\mu}}_l \preceq \hat{\boldsymbol{\mu}}_k \quad \Longleftrightarrow \quad l \geq k \qquad \hat{\boldsymbol{\nu}}_l \preceq \hat{\boldsymbol{\nu}}_k, \quad \hat{\boldsymbol{\tau}}_l \preceq \hat{\boldsymbol{\tau}}_k \quad \Longleftrightarrow \quad l \leq k$$

- bounds among properties: $\hat{\boldsymbol{\mu}}_k \preceq \hat{\boldsymbol{\nu}}_{n+1-k}$, $\hat{\boldsymbol{\nu}}_k \preceq \hat{\boldsymbol{\mu}}_{\lceil n/k \rceil}$, from

$$\lceil n/w \rceil \leq h \leq n - w + 1 \qquad \lceil n/h \rceil \leq w \leq n - h + 1$$

- duality

$$h(\hat{\xi}^\dagger) = w(\hat{\xi}), \quad w(\hat{\xi}^\dagger) = h(\hat{\xi}), \quad r(\hat{\xi}^\dagger) = -r(\hat{\xi}),$$

k-partitionability, k-producibility and k-stretchability

height, width and rank of a Young diagram $\Longrightarrow$ properties

$$h(\hat{\xi}) := |\hat{\xi}|$$
$$w(\hat{\xi}) := \max \hat{\xi}$$
$$r(\hat{\xi}) := w(\hat{\xi}) - h(\hat{\xi})$$

- monotones

$$\hat{\upsilon} \prec \hat{\xi} \quad \Longrightarrow \quad h(\hat{\upsilon}) >$$

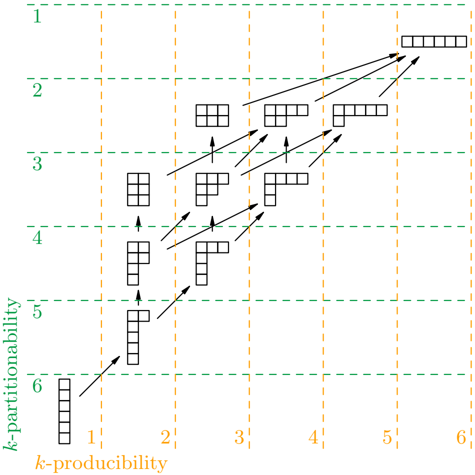
- chains

$$\hat{\boldsymbol{\mu}}_l \preceq \hat{\boldsymbol{\mu}}_k \quad \Longleftrightarrow \quad l \geq k$$

- bounds among properties: $\hat{\boldsymbol{\mu}}$

$$\lceil n/w \rceil \leq h \leq n - w -$$

- duality



$$h(\hat{\xi}^\dagger) = w(\hat{\xi}), \quad w(\hat{\xi}^\dagger) = h(\hat{\xi}), \quad r(\hat{\xi}^\dagger) = -r(\hat{\xi}),$$

Take home message

Notions of correlations

- *pure states* of classical systems are uncorrelated (product)
- correlation in *pure states* is of quantum origin,
  this is what we call entanglement
- *mixed states:* uncorrelated/correlated;
  separable/entangled, if it can/cannot be mixed from uncorrelated ones

Correlation measures

- correlation: "how correlated = how not uncorrelated"
- *pure states:* entanglement = correlation,
  mixed states: e.g., average entanglement of the optimal decomp.

These principles were applied for multipartite systems painlessly

- general case: partitions, three-level structure
- no convex hull for correlation: simpler classification
- permutation invariant case: Young diagrams, conjugation
- partitionability/producibility/stretchability: height/width/rank

Thank you for your attention!

Szalay, Quantum **3**, 204 (2019)

Szalay, Barcza, Szilvási, Veis, Legeza, SciRep **7**, 2237 (2017)

Szalay, PRA **92**, 042329 (2015)

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MINISTRY OF HUMAN CAPACITIES

NATIONAL RESEARCH, DEVELOPMENT AND INNOVATION OFFICE HUNGARY

PROJECT FINANCED FROM THE NRDI FUND

MOMENTUM OF INNOVATION

Correlation-based clustering – Overview

Bipartite correlation clustering (for treshold T_b): split $\gamma = C_1|C_2|\dots|C_{|\gamma|}$, the connectivity clustering of the graph $\big(L, \{(i,j)\}_{T_\mathrm{b}\leq C_{i|j}}\big)$,

Multipartite correlation clustering:
give a split $\beta = B_1|B_2|\dots|B_{|\beta|}$, *if exists*, for which

- the subsystems $B \in \beta$ are weakly correlated with one another C_β low
- the elementary subsystems $\{i\} \subseteq B$ are strongly correlated with one another $C_{k\text{-part},B}$, $C_{k\text{-prod},B}$ high

Problems

- hidden correlation: $\gamma \prec \beta$
- hard to find β, too many possibilities to check
- meaning/definition of "C_β low" and "$C_{k\text{-part},B}$, $C_{k\text{-prod},B}$ high"

We have a method to handle these.

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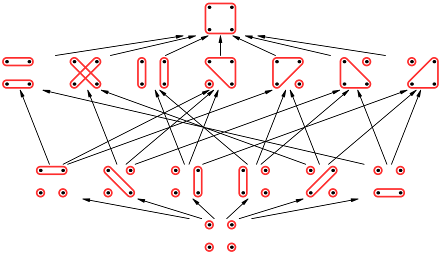
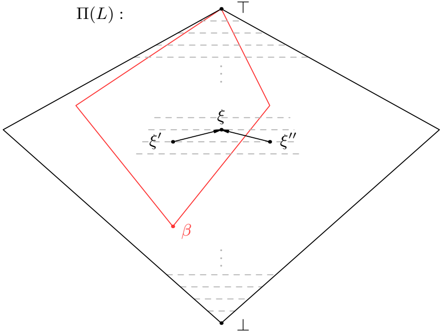
Correlation-based clustering – Definition

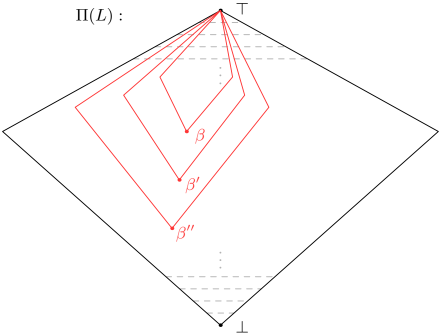
- multipartite monotonity: $\upsilon \preceq \xi \Leftrightarrow C_\upsilon \geq C_\xi$
- covering (being neighbours): $\xi' \prec\!\!\cdot\; \xi$
- derivative: $C_{\xi'}(\rho_L) - C_\xi(\rho_L) = C_{\xi'\setminus\xi}(\rho_{X_*})$
- reformulation: $\exists T_\mathrm{m} > 0$, such that
  $\forall \xi, \xi' \in \Pi(L)$ such that $\xi' \prec\!\!\cdot\; \xi$, and $\beta \preceq \xi$, then
  $\beta \preceq \xi' \;\Leftrightarrow\; C_{\xi'}(\rho_L) - C_\xi(\rho_L) \leq T_\mathrm{m}$

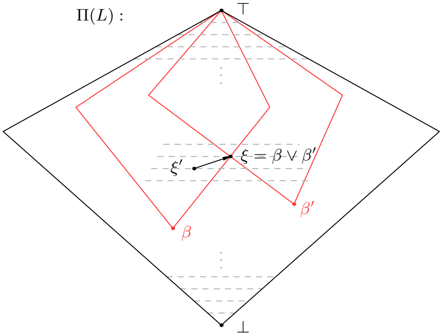




Correlation-based clustering – Properties

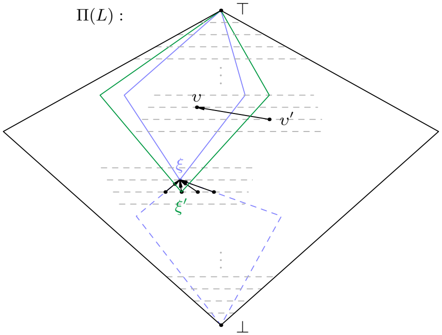
- there might not exist such clustering
- there may exist compatible clusterings (of different T_{m}s),
  but there exist no contradictory ones:

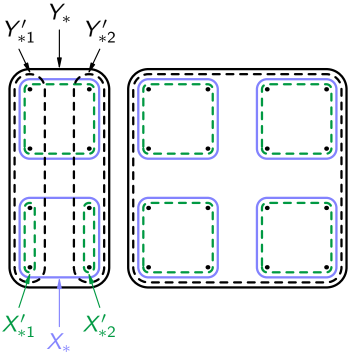




Correlation-based clustering – Finding β

- successive refinement from $\top$ to $\bot$ (taking the smallest step):
  $\forall \upsilon, \upsilon' \in \Pi(L)$ s.t. $\upsilon' \prec \upsilon$, and $\forall \xi \in \Pi(L)$ s.t. $\xi \preceq \upsilon$ but $\xi \npreceq \upsilon'$,
  then $\min_{\xi' \prec \xi} C_{\xi'}(\rho_L) - C_\xi(\rho_L) \leq C_{\upsilon'}(\rho_L) - C_\upsilon(\rho_L)$
- hint: does not dissect $G \in \gamma$ (bipart. corr. clustering), since
  $T_\mathrm{b} \leq C_{\xi'}(\rho_L) - C_\xi(\rho_L)$ if ξ does not dissect G while ξ' does
- hidden correlation: $\gamma \prec \beta$



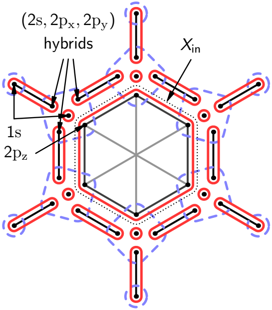


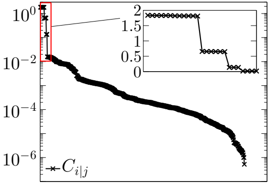
Example: Electron system of molecules

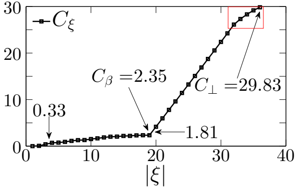
benzene (C_6H_6)

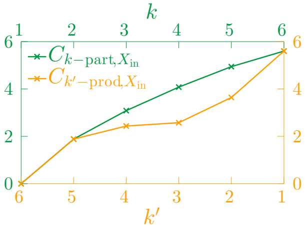
$C_\alpha = 29.52$

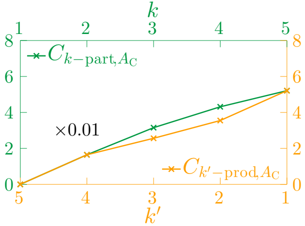
$C_\beta = 2.33$











(in units ln 4)

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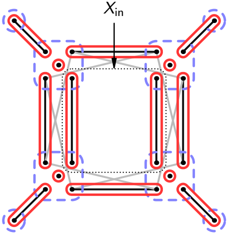
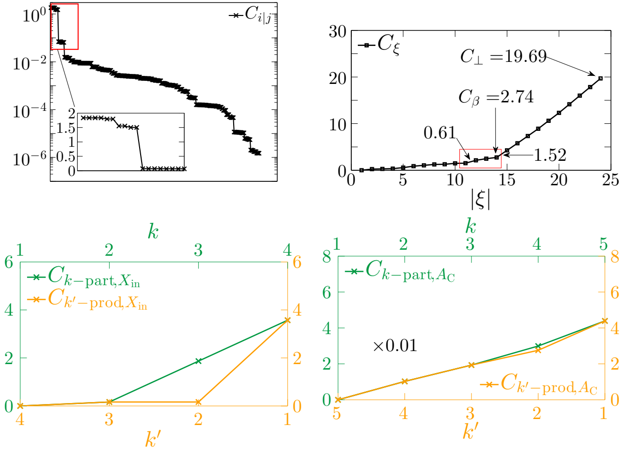
Example: Electron system of molecules

cyclobutadiene (C_4H_4)

$C_\alpha = 19.48$

$C_\beta = 3.17$





(in units $\ln 4$)

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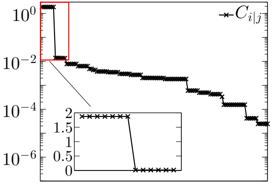
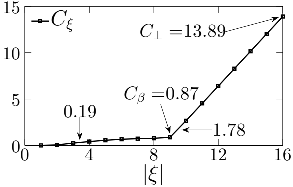
Example: Electron system of molecules

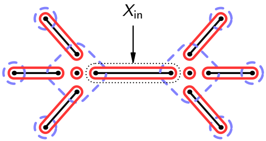
ethane (C_2H_6)

$C_\alpha = 13.84$

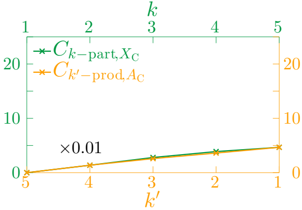
$C_\beta = 0.90$







$$C_{\text{2-part},X_{\text{in}}} = C_{\text{1-prod},X_{\text{in}}} = C_{\perp,X_{\text{in}}} = 1.796$$



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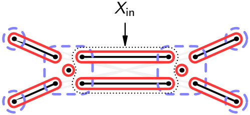
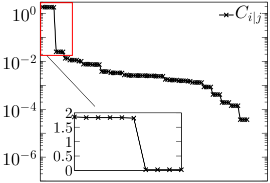
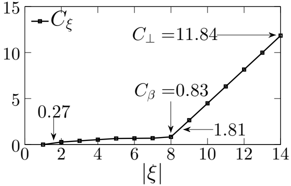
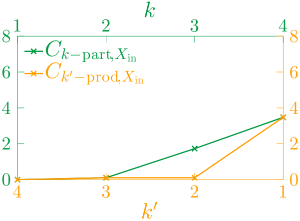
Example: Electron system of molecules

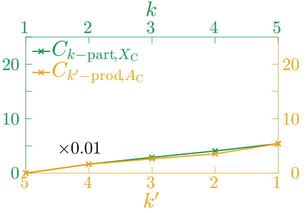
ethylene (C_2H_4)
$C_\alpha = 11.76$
$C_\beta = 1.00$











(in units ln 4)

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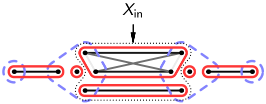
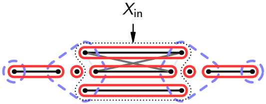
Example: Electron system of molecules

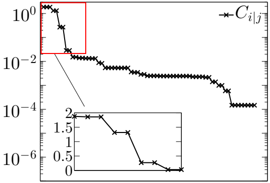
acetylene (C_2H_2)

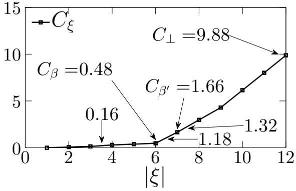
$C_\alpha = 9.74$

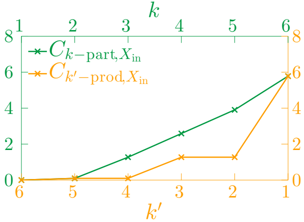
$C_\beta = 0.45, 1.30$

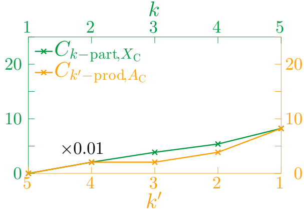












(in units $\ln 4$)

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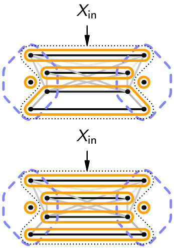
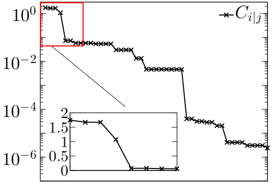
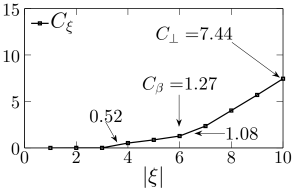
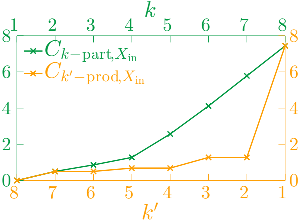
Example: Electron system of molecules

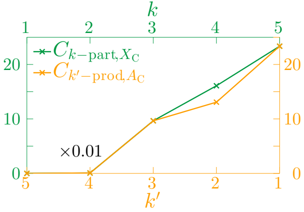
dicarbon (C_2H_0)
$C_\alpha = 7.06$
$C_\beta = 0.89, 1.51$











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