Metrological entanglement criteria

Szilárd Szalay and Géza Tóth

Wigner Research Centre for Physics, Budapest, Hungary
University of the Basque Country UPV/EHU, Bilbao, Spain

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Overview

- multipartite entanglement:
  classification / qualification / quantification

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- partial separability / criteria / entropic measures

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$$D(\rho) \quad \text{(prod.)}$$

$$\vee\!|$$

$$D_{\mathsf{avg}}(\rho) \quad \text{(avg.)}$$

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*"Alternatives of entanglement depth
and metrological entanglement criteria"*

Szalay, Tóth, arXiv:2408.15350 **[quant-ph]** (2024), under review in Quantum

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- estimation of parameter θ in the dynamics $U(\theta) = e^{-iA\theta}$
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- qantum Fisher information is the convex roof of the variance
- convex bound by the convex roof of the original bound

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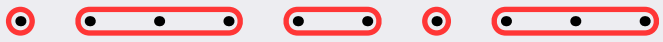
2.4				3

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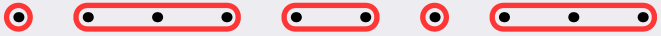
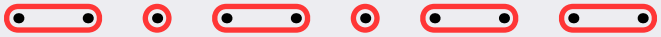
2.4		3
2.2		2

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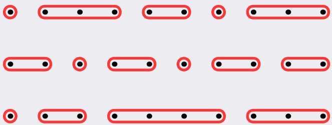
2.2		2
3		4



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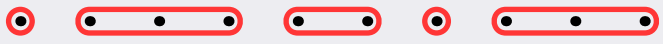
2.4		3
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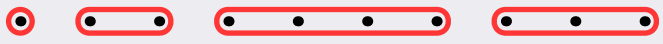
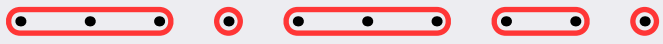
3		4
2.4		3

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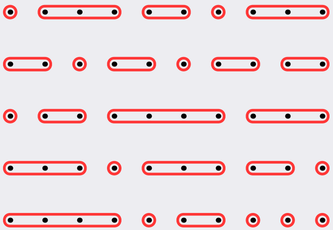
2.2		2
3		4
2.4		3
2.4		4
⋮	⋮	⋮



$\mathrm{in}_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$

$\mathrm{in}_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$

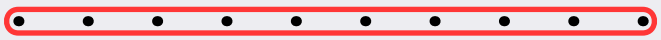
$\{(p_j,\pi_j)\}\vdash\rho \max_j D_{\mathrm{avg}}(\pi_j)$

$\{(p_j,\pi_j)\}\vdash\rho \sum_j p_j D_{\mathrm{avg}}(\pi_j)$

The meaning of the quantities

$$\begin{array}{ccccccc} & & D^{\text{oF}}(\rho) & \leq & D(\rho) & & \text{(prod.)} \\ & & \vee\!| & & \vee\!| & & \\ F_{\text{Q}}(\rho, J^z)/n & \leq & D^{\text{oF}}_{\text{avg}}(\rho) & \leq & D_{\text{avg}}(\rho) & & \text{(avg.)} \end{array}$$

examples

$D_{\text{avg}}(\pi_j)$	$\longleftarrow$	π_j	$\longmapsto$	$D(\pi_j)$
10				10

$\text{in}_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$

$D \quad (\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$

- avg-entanglement depth

$$D_{\text{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\text{avg}}(\pi_j)$$

- avg-entanglement depth of formation

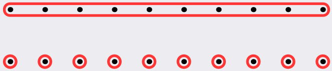
$$D^{\text{oF}}_{\text{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D_{\text{avg}}(\pi_j)$$

average size of entangled subsystems (ASES)

The meaning of the quantities

$$
\begin{array}{ccccccc}
 & & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & & \text{(prod.)} \\
 & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
F_{\mathrm{Q}}(\rho, J^z)/n & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & \text{(avg.)}
\end{array}
$$

examples

$D_{\mathrm{avg}}(\pi_j)$	$\longleftarrow$ π_j $\longmapsto$	$D(\pi_j)$
10		10
1		1

$\ldots\mathrm{in}_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$

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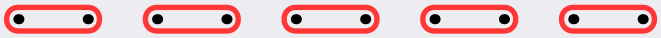
\end{array}
$$

examples

$D_{\mathrm{avg}}(\pi_j)$	$\longleftarrow$	π_j	$\longmapsto$	$D(\pi_j)$
2				2

$\ldots\mathrm{in}_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$

$\ldots(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$

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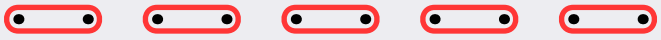
$$D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D_{\mathrm{avg}}(\pi_j)$$

average size of entangled subsystems (ASES)

$$\begin{array}{ccccccc} & & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & & (\text{prod.}) \\ & & \vee\!| & & \vee\!| & & \\ F_Q(\rho, J^z)/n & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & (\text{avg.}) \end{array}$$

examples

$D_{\mathrm{avg}}(\pi_j)$	$\longleftarrow$ π_j $\longmapsto$	$D(\pi_j)$
2		2
1.2		2

$\text{in}_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$

$\text{in}_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$

$$D_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\mathrm{avg}}(\pi_j)$$

- avg-entanglement depth of formation

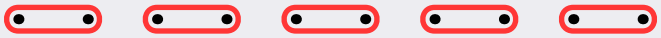
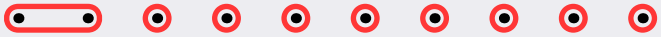
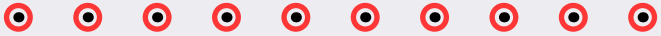
$$D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D_{\mathrm{avg}}(\pi_j)$$

average size of entangled subsystems (ASES)

The meaning of the quantities

$$
\begin{array}{ccccccc}
 & & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & & \text{(prod.)} \\
 & & \vee\!| & & \vee\!| & & \\
F_{\mathrm{Q}}(\rho, J^z)/n & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & \text{(avg.)}
\end{array}
$$

examples

$D_{\mathrm{avg}}(\pi_j)$	$\longleftarrow$ π_j $\longmapsto$	$D(\pi_j)$
2		2
1.2		2
1		1

$\mathrm{in}_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$

$\mathrm{in}_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$

$_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\mathrm{avg}}(\pi_j)$

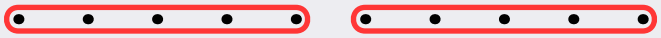
- avg-entanglement depth of formation

$$D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D_{\mathrm{avg}}(\pi_j)$$

average size of entangled subsystems (ASES)

$$
\begin{array}{ccccccc}
 & & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & & \text{(prod.)} \\
 & & \vee\!| & & \vee\!| & & \\
F_{\mathrm{Q}}(\rho, J^z)/n & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & \text{(avg.)}
\end{array}
$$

examples

$D_{\mathrm{avg}}(\pi_j)$	$\longleftarrow$	π_j	$\longmapsto$	$D(\pi_j)$
5				5

$\text{in}_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$

$(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$

- avg-entanglement depth

$$D_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\mathrm{avg}}(\pi_j)$$

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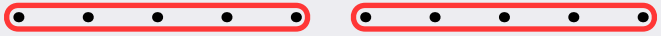
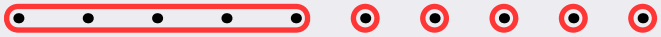
$$D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D_{\mathrm{avg}}(\pi_j)$$

average size of entangled subsystems (ASES)

$$\begin{array}{ccccc} & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & \text{(prod.)} \\ & \vee\!| & & \vee\!| & \\ F_{\mathrm{Q}}(\rho, J^z)/n \;\leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & \text{(avg.)} \end{array}$$

examples

$D_{\mathrm{avg}}(\pi_j)$	$\longleftarrow$ π_j $\longmapsto$	$D(\pi_j)$
5		5
3		5

$\mathrm{in}_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$

$\mathrm{in}_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$

$$D_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\mathrm{avg}}(\pi_j)$$

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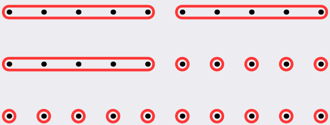
$$D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D_{\mathrm{avg}}(\pi_j)$$

average size of entangled subsystems (ASES)

$$\begin{array}{ccccccc} & & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & & (\text{prod.}) \\ & & \vee\!| & & \vee\!| & & \\ F_{\mathrm{Q}}(\rho, J^z)/n & \leq & D_{\mathrm{avg}}^{\mathrm{oF}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & (\text{avg.}) \end{array}$$

examples



$D_{\mathrm{avg}}(\pi_j)$	$\longleftarrow$ π_j $\longmapsto$	$D(\pi_j)$
5		5
3		5
1		1

$\text{in}_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$

$\text{in}_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$

$_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\mathrm{avg}}(\pi_j)$

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average size of entangled subsystems (ASES)

weaker bound $F_{\mathrm{Q}}(\rho, J^{\mathrm{z}})/n \leq D(\rho)$

- $\rho_\epsilon := \epsilon\pi_k + (1-\epsilon)\rho_1$ for $\epsilon > 0$
  with $\pi_k := |\psi_k\rangle\langle\psi_k| \in \mathcal{C}_{k\text{-prod}}$ and $\rho_1 \in \mathcal{C}_{1\text{-prod}}$, $\mathrm{Tr}(\pi_k\rho_1) = 0$

Convex vs. original: $F_\mathrm{Q}(\rho, J^\mathrm{z})/n \leq D^{\mathrm{oF}}(\rho) \leq D(\rho)$

weaker bound $F_\mathrm{Q}(\rho, J^\mathrm{z})/n \leq D(\rho)$

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- ρ_ϵ is not k'-producible for $k' < k$, so $D(\rho_\epsilon) = k$

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weaker bound $F_{\mathrm{Q}}(\rho, J^{\mathrm{z}})/n \leq D(\rho)$

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  with $\pi_k := |\psi_k\rangle\langle\psi_k| \in \mathcal{C}_{k\text{-prod}}$ and $\rho_1 \in \mathcal{C}_{1\text{-prod}}$, $\mathrm{Tr}(\pi_k\rho_1) = 0$
- ρ_ϵ is not k'-producible for $k' < k$, so $D(\rho_\epsilon) = k$
- ρ_ϵ is much less entangled as π_k itself, a much lower F_{Q}/n is expected

Convex vs. original: $F_Q(\rho, J^z)/n \le D^{oF}(\rho) \le D(\rho)$

stronger bound $F_Q(\rho, J^z)/n \le D^{oF}(\rho)$

- for all pure decompositions $\rho = \sum_j p_j \pi_j$, with $q_k = \sum_{j:D(\pi_j)=k} p_j$

$$F_Q(\rho, J^z)/n \le D^{oF}(\rho) \le \sum_j p_j D(\pi_j)$$

$$= \sum_{k=1}^{n} q_k \sum_{j:D(\pi_j)=k} \frac{p_j}{q_k} D(\pi_j) = \sum_{k=1}^{n} q_k k$$

Convex vs. original: $F_Q(\rho, J^z)/n \le D^{oF}(\rho) \le D(\rho)$

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- e.g., let $n = 10$, $F_Q(\rho, J^z) \ge 30$, $F_Q(\rho, J^z)/n \ge 3$
- always exists k-producible π_j for at least one $k \ge 3$

Convex vs. original: $F_\mathrm{Q}(\rho, J^z)/n \leq D^{\mathrm{oF}}(\rho) \leq D(\rho)$

stronger bound $F_\mathrm{Q}(\rho, J^z)/n \leq D^{\mathrm{oF}}(\rho)$

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- if there are k-producible pure states π_j for $k < 3$, then this has to be compensated by k-producible π_j-s for $k > 3$

stronger bound $F_Q(\rho, J^z)/n \le D^{oF}(\rho)$

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$$F_Q(\rho, J^z)/n \le D^{oF}(\rho) \le \sum_j p_j D(\pi_j)$$
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- at least $2q_1$ weight of 4-producible states are needed for this
  $3 \le 1q_1 + 3q_3 + 4q_4 = q_1 + 3(1 - q_1 - q_4) + 4q_4$ leads to $2q_1 \le q_4$

stronger bound $F_{\mathrm{Q}}(\rho, J^z)/n \leq D^{\mathrm{oF}}(\rho)$

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$$F_{\mathrm{Q}}(\rho, J^z)/n \leq D^{\mathrm{oF}}(\rho) \leq \sum_j p_j D(\pi_j)$$
$$= \sum_{k=1}^{n} q_k \sum_{j:D(\pi_j)=k} \frac{p_j}{q_k} D(\pi_j) = \sum_{k=1}^{n} q_k k$$

- e.g., let $n = 10$, $F_{\mathrm{Q}}(\rho, J^z) \geq 30$, $F_{\mathrm{Q}}(\rho, J^z)/n \geq 3$
- always exists k-producible π_j for at least one $k \geq 3$
- if there are k-producible pure states π_j for $k < 3$,
  then this has to be compensated by k-producible π_j-s for $k > 3$
- at least $2q_1$ weight of 4-producible states are needed for this
  $3 \leq 1q_1 + 3q_3 + 4q_4 = q_1 + 3(1 - q_1 - q_4) + 4q_4$ leads to $2q_1 \leq q_4$
- or at least the same q_1 weight of 5-producible states
  $3 \leq 1q_1 + 3q_3 + 5q_5 = q_1 + 3(1 - q_1 - q_5) + 5q_5$ leads to $q_1 \leq q_5$

Take home message

- k-producibility, k-average: one-parameter partial separability properties

Take home message

- *k*-producibility, *k*-average: one-parameter partial separability properties
- partitionability, producibility, stretchability,
  squareability, toughness, size-Rényi/Tsallis, Dim, DoF. . .

Take home message

- k-producibility, k-average: one-parameter partial separability properties
- partitionability, producibility, stretchability, squareability, toughness, size-Rényi/Tsallis, Dim, DoF...
- properties naturally characterized by depth and depth of formation

Take home message

- *k*-producibility, *k*-average: one-parameter partial separability properties
- partitionability, producibility, stretchability,
  squareability, toughness, size-Rényi/Tsallis, Dim, DoF...
- properties naturally characterized by depth and depth of formation
- metrological entanglement criteria

  metrological precision (by quantum Fisher information)

  vs. multipartite entanglement (by entanglement depths)

Take home message

- k-producibility, k-average: one-parameter partial separability properties
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metrological precision (by quantum Fisher information)
vs. multipartite entanglement (by entanglement depths)

$$
\begin{array}{ccccccc}
 & & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & & \text{(prod.)} \\
 & & \vee\!| & & \vee\!| & & \\
\frac{1}{n}F_{\mathrm{Q}}(\rho, J^{\mathrm{z}}) & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & \text{(avg.)}
\end{array}
$$

Thank you for your attention!

"Alternatives of entanglement depth
and metrological entanglement criteria"

Szalay, Tóth, arXiv:2408.15350 **[quant-ph]** (2024), under review in Quantum

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