Dynamical features of iterated nonlinear qubit protocols with measurement selection

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Collaboration:
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Quantum theory: linear or nonlinear?

1., Closed systems - unitary operators - linear evolution

2., Quantum channels - completely positive maps - linear evolution

If quantum states evolved nonlinearly

- then hard problems (NP complete)
  would be easily solved (in polynomial time)

  D. S. Abrams and S. Lloyd, Phys. Rev. Lett. 81, 3992 (1998)
- quick discrimination of nonorthogonal states - generic feature

  A. M. Childs and J. Young, Phys. Rev. A, 93, 022314 (2016)
- BEC + nonlinear quantum walks

  D. A. Meyer and T. G. Wong, New J. Phys. 15, 063014 (2013)

  A. Alberti and S. Wimberger, Phys. Rev. A 96, 023620 (2017)

  M. Maeda, H. Sasaki, E. Segawa, et al. Quantum. Inf. Process. 17, 215 (2018)

Effective nonlinearity in quantum theory

PHYSICAL REVIEW A **89**, 012312 (2014)

Quantum search with general nonlinearities

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(Received 13 November 2013; published 14 January 2014)

Evolution by the Gross-Pitaevskii equation, which describes Bose-Einstein condensates under certain conditions, solves the unstructured search problem more efficiently than does the Schrödinger equation, because it includes a cubic nonlinearity, proportional to $|\psi|^2\psi$. This is not the only nonlinearity of the form $f(|\psi|^2)\psi$ that arises in effective equations for the evolution of real quantum physical systems, however: The cubic-quintic nonlinear Schrödinger equation describes light propagation in nonlinear Kerr media with defocusing corrections, and the logarithmic nonlinear Schrödinger equation describes Bose liquids under certain conditions. Analysis of computation with such systems yields some surprising results; for example, when time-measurement precision is included in the resource accounting, searching a "database" when there is a single correct answer may be easier than searching when there are multiple correct answers. In each of these cases the nonlinear equation is an effective approximation to a multiparticle Schrödinger equation, for search by which Grover's algorithm is optimal. Thus our results lead to quantum information-theoretic bounds on the physical resources required for these effective nonlinear theories to hold, asymptotically.

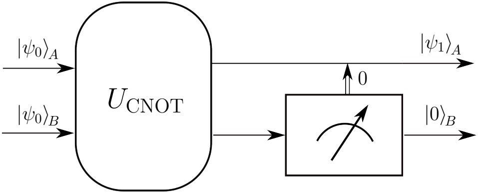
Nonlinear transformations by selective evolution

3., Measurements

- ✓ projection (von Neumann)
- ✓ probabilistic (Born)
- ✓ information gained
- ☞ information feed-back, post-selection

↯ breaking linearity ↯

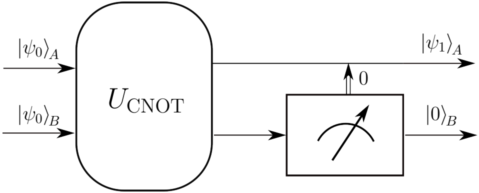
Transformation of a qubit



$$|\psi_0\rangle = \frac{1}{\sqrt{1+|z|^2}}\left(|0\rangle + z\,|1\rangle\right)$$

H. Bechmann-Pasquinucci et al. Phys. Lett. A **242**, 198 (1998).

Transformation of a qubit



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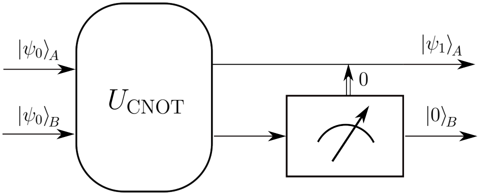
$$\left|\Psi^{\rm in}\right\rangle_{AB} = |\psi_0\rangle_A \otimes |\psi_0\rangle_B = \frac{1}{1+|z|^2}\left(|00\rangle + z\,|01\rangle + z\,|10\rangle + z^2\,|11\rangle\right)$$

Transformation of a qubit



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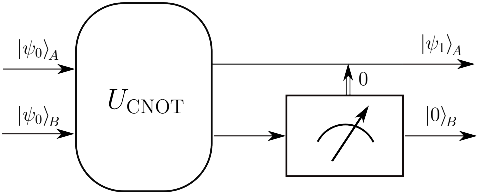
Transformation of a qubit



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H. Bechmann-Pasquinucci et al. Phys. Lett. A **242**, 198 (1998).

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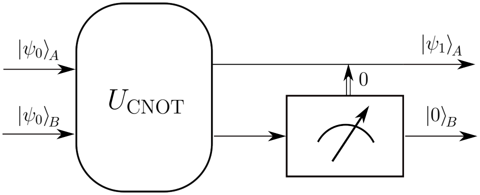
- after projecting qubit B to $|0\rangle$:

$$|\psi_1\rangle_A = \frac{1}{\sqrt{1+|z|^2}} \left(|0\rangle + z^2\,|1\rangle\right)$$

Transformation of a qubit



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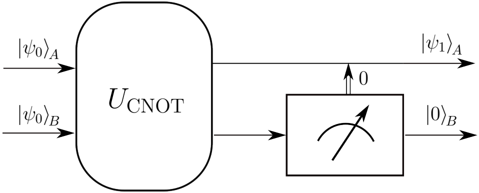
- after projecting qubit B to $|0\rangle$:

$$|\psi_1\rangle_A = \frac{1}{\sqrt{1+|z|^2}}\left(|0\rangle + z^2\,|1\rangle\right) \quad \longrightarrow \quad f(z) = z^2$$

Transformation of a qubit

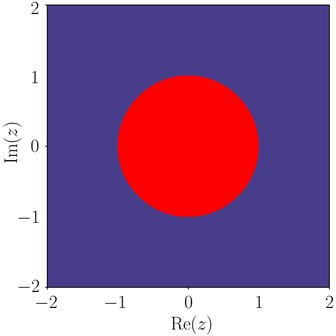


$$|\psi_0\rangle = \frac{1}{\sqrt{1+|z|^2}}\left(|0\rangle + z\,|1\rangle\right)$$

$$\downarrow$$

$$|\psi_1\rangle = \frac{1}{\sqrt{1+|z|^4}}\left(|0\rangle + z^2\,|1\rangle\right)$$

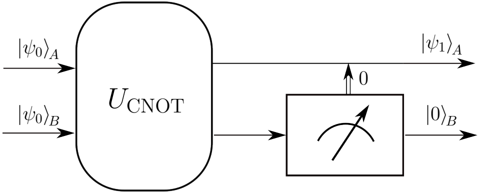
H. Bechmann-Pasquinucci et al. Phys. Lett. A **242**, 198 (1998).



Iteration of $f(z) = z^2$ (complex plane)

- $|z| < 1 \rightarrow 0$ (stable fixed point)
- $|z| > 1 \rightarrow \infty$ (stable fixed point)
- $|z| = 1 \rightarrow$ **no convergence**

Transformation of a qubit



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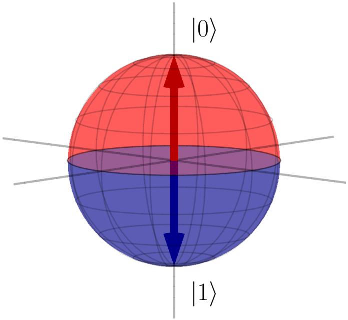


Iteration of $f(z) = z^2$
(Bloch sphere)

- ☞ $|z| < 1$ states converge to $|0\rangle$
- ☞ $|z| > 1$ states converge to $|1\rangle$
- ☞ $z = 1$ **weird points: the Julia set**

Iterative nonlinear quantum protocols

- ensemble of qubits - quantum gates - measurement
- all basic components of a universal quantum circuit
- iterate the protocol

Complex chaos

- deterministic
- pure quantum states remain pure
- positive Lyapunov exponent

T. Kiss, I. Jex, G. Alber, S. Vymětal, Phys. Rev. A **74**, 040301(R) (2006).

Iterative nonlinear quantum protocols

- Ensemble of qubits in *pure state* $|\psi_0\rangle \sim |0\rangle + z\,|1\rangle \quad (z \in \mathbb{C})$
  1. Take them pairwise: $|\Psi_0\rangle = |\psi_0\rangle_A \otimes |\psi_0\rangle_B$
  2. Apply an **entangling** two-qubit operation U
  3. Measure the state of qubit B — keep A only for result 0
- Smaller ensemble in *pure state* $|\psi_1\rangle \sim |0\rangle + f(z)\,|1\rangle$
- Quantum magnification bound: exponential downscaling of the ensemble

$$U \leftrightarrow f(z) = \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2}$$

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016)

Historical remarks on complex dynamics

- Iterated rational polynomials: $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \quad f^{\circ n} \to ?$
- One century of complex chaos:

  1871 idea of iterated functions by Ernst Schröder

  Ueber iterirte Functionen., Math. Ann.

  1906 first weird example by P. Fatou: $z \mapsto z^2/(z^2+2)$

  1920ies G. Julia, S. Lattès, & ...

  1970ies Computers help visualize: B. Mandelbrot & ...

- A good book:
  J.W. Milnor *Dynamics in One Complex Variable*, (Vieweg, 2000)

Iterative dynamics - examples

CNOT gate plus a single qubit gate

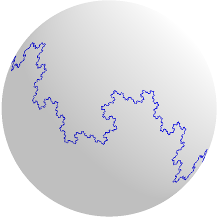
$$U = \begin{pmatrix} \cos\theta & \sin\theta\, e^{i\varphi} \\ -\sin\theta\, e^{-i\varphi} & \cos\theta \end{pmatrix}$$

Family of maps over $\hat{\mathbb{C}}$:

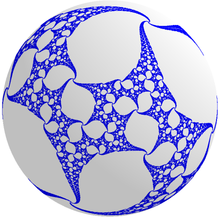
$$z \mapsto f_p(z) = \frac{z^2 + p}{1 - p^* z^2} \qquad p = \tan\theta\, e^{i\varphi}$$

$p \in \mathbb{C}$ parameter of the gate

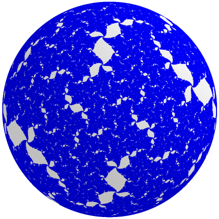
Iterative dynamics - Julia sets on the Bloch sphere



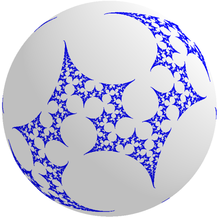
(a) $\theta = 0.4, \varphi = \frac{\pi}{2}$



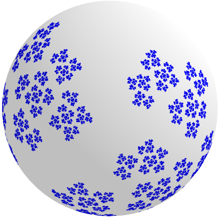
(b) $\theta = 0.55, \varphi = \frac{\pi}{2}$



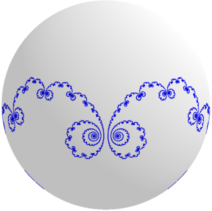
(c) $\theta = 0.633, \varphi = \frac{\pi}{2}$



(d) $\theta = 1.05, \varphi = \frac{\pi}{2}$



(e) $\theta = 0.5, \varphi = 0.5$

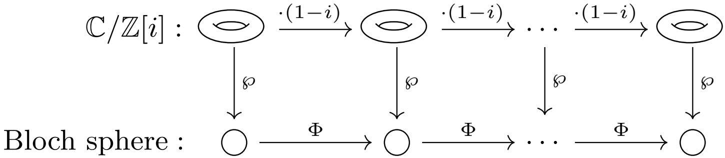


(f) $\theta = 0.232, \varphi = 0$

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

Lattès map: $J = \hat{\mathbb{C}}$

$$f(z) = \frac{z^2 + i}{iz^2 + 1}, \quad p = i$$



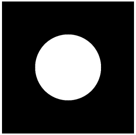
A commutative diagram:

- map on the Bloch sphere $\leftrightarrow \times(1-i)^n$ on the torus
- all initial states are weird
- ergodicity

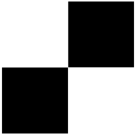
Lattès, S (1918), Les Comptes rendus de l'Académie des sciences, 166: 26-28

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

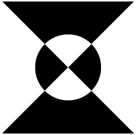
Lattès map: ergodic dynamics



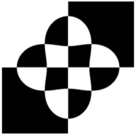
(a) $|z| > 1$



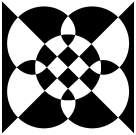
(b) $|f(z)| > 1$



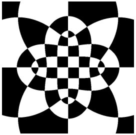
(c) $|f^{\circ 2}(z)| > 1$



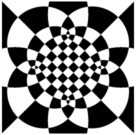
(d) $|f^{\circ 3}(z)| > 1$



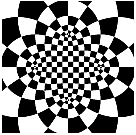
(e) $|f^{\circ 4}(z)| > 1$



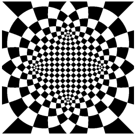
(f) $|f^{\circ 5}(z)| > 1$



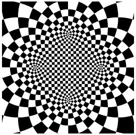
(g) $|f^{\circ 6}(z)| > 1$



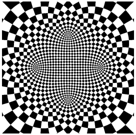
(h) $|f^{\circ 7}(z)| > 1$



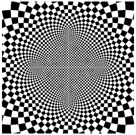
(i) $|f^{\circ 8}(z)| > 1$



(j) $|f^{\circ 9}(z)| > 1$



(k)$|f^{\circ 10}(z)| > 1$



(l) $|f^{\circ 11}(z)| > 1$

Lattès map with noisy initial states

Dynamics represented by $\mathbb{R}^3 \to \mathbb{R}^3$ functions:

$$u' = \frac{u^2 - v^2}{1 + w^2}, \quad v' = \frac{2w}{1 + w^2}, \quad w' = -\frac{2uv}{1 + w^2}$$

No book by Milnor! :-(

Asymptotics: all mixed initial states $\to$ completely mixed state

O. Kálmán, T. Kiss and I. Jex, J Russ Laser Res 39: 382 (2018)

CNOT + Hadamard gate: phase transition

Noisy (mixed) initial states:

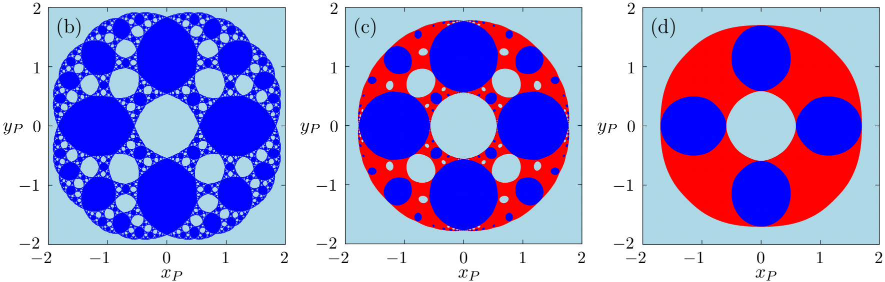
$$\rho \xrightarrow{\mathcal{M}} \rho' = U_H \, \frac{\rho \odot \rho}{\mathrm{Tr}(\rho \odot \rho)} \, U_H^\dagger$$

where

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \rho = \frac{1}{2} \begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix}$$

Purity: $P = \mathrm{Tr}(\rho^2) = (1 + u^2 + v^2 + w^2)/2 \leq 1$

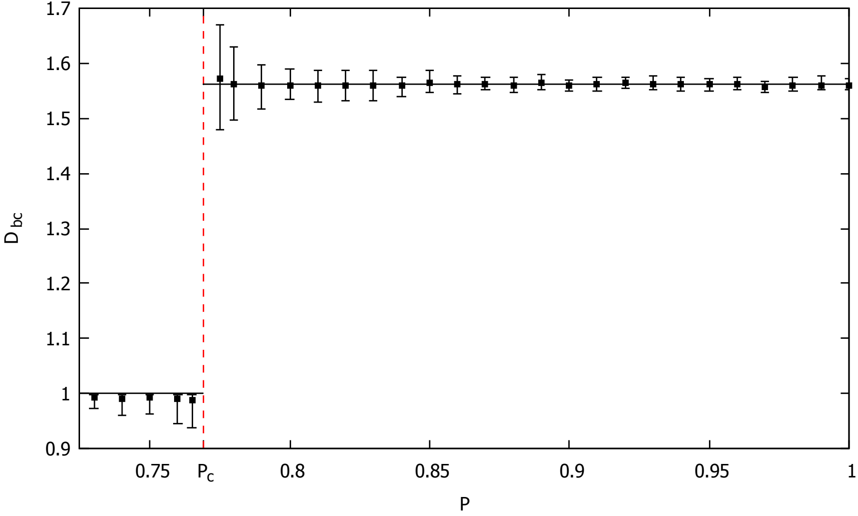
Convergence for different purities P



$P = 1$ $P = 0.87$ $P = 0.75$

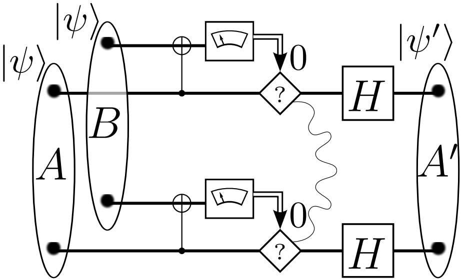
Light blue: convergence to $|0\rangle$ after an even number of steps
Dark blue: convergence to $|0\rangle$ after an odd number of steps
Red: convergence to the completely mixed state

Fractal dimension D_{bc} as a function of purity P



M. Malachov, I. Jex, O. Kálmán, and T. Kiss, Chaos 29, 033107 (2019)

LOCC scheme with 2 qubits



$$|\psi\rangle = c_1|00\rangle + c_2|01\rangle + c_3|10\rangle + c_4|11\rangle$$

$$|\psi'\rangle = U_H \otimes U_H \left[\mathcal{N}(c_1^2|00\rangle + c_2^2|01\rangle + c_3^2|10\rangle + c_4^2|11\rangle)\right]$$

2 qubits: chaotic entanglement

Asymptotic states

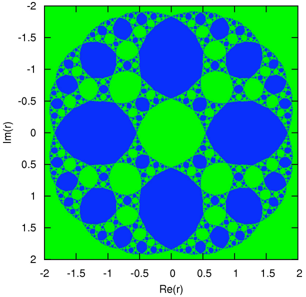
Green: Fully entangled:

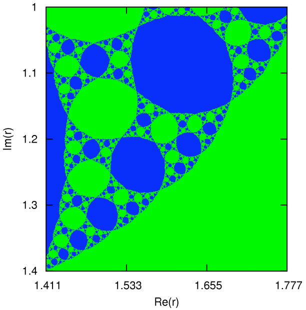
$$|\psi^{(\infty)}\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle\right)$$

Blue: Completely separable, oscillatory:

$$|\psi^{(\infty)}\rangle \to \left\{ |00\rangle \, , \, \frac{1}{2} \left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right) \right\}$$

T. Kiss, S. Vymětal, L. D. Tóth, A. Gábris,
I. Jex, G. Alber, PRL **107**, 100501 (2011)





An application for state orthogonalization

- nonlinear transformation: $f_{\varphi=0} = \dfrac{2z}{1+z^2}$

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

An application for state orthogonalization

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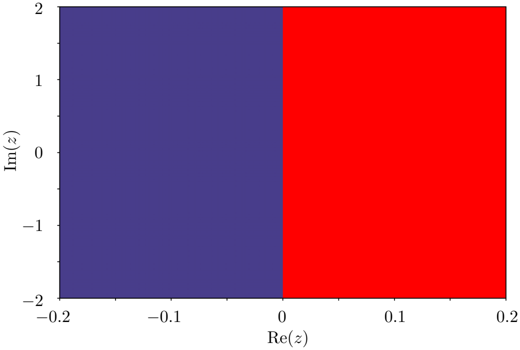

- two superattractive fixed points: 1 and -1
- Julia set: imaginary axis

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

An application for state orthogonalization

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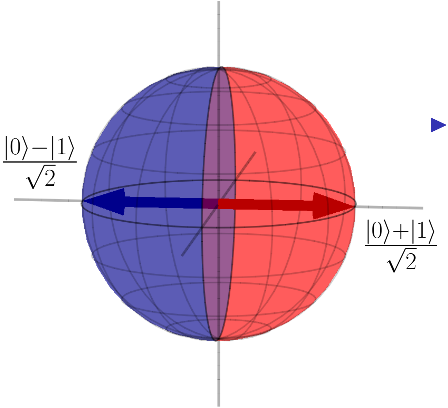


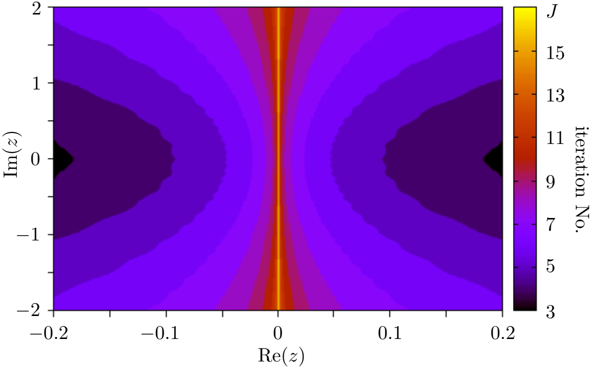
- Julia set: longitudinal great circle through y axis
  - equally separates regions of convergence

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

An application for state orthogonalization

- nonlinear transformation: $f_{\varphi=0} = \dfrac{2z}{1+z^2}$

- From highly overlapping to almost orthogonal in only 3 steps



$$|\Psi_0\rangle_I = \frac{|0\rangle + 0.2\,|1\rangle}{\sqrt{1+(0.2)^2}}$$

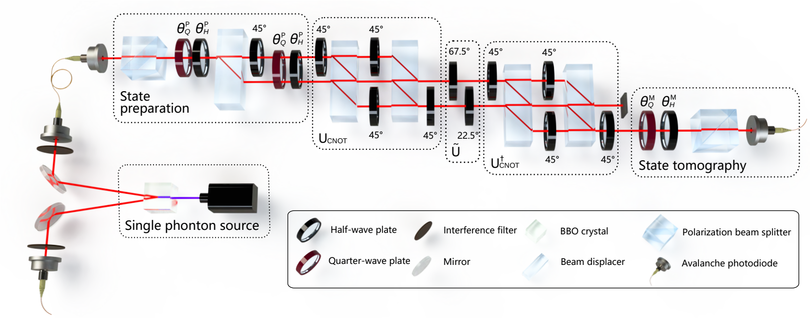
$$|\Psi_0\rangle_{II} = \frac{|0\rangle - 0.2\,|1\rangle}{\sqrt{1+(0.2)^2}}$$

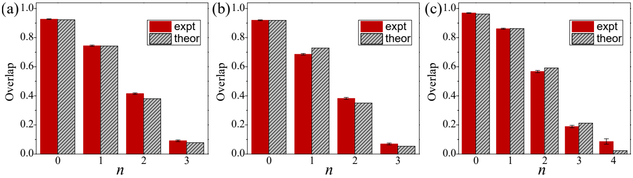
$${}_I\langle\Psi_0\,|\,\Psi_0\rangle_{II} \approx 0.92$$

$${}_I\langle\Psi_3\,|\,\Psi_3\rangle_{II} \approx 0.08$$

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

Quantum state orthogonalization: experiment

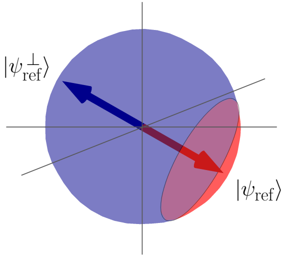




a) $\{z_1 = 0.2, z_2 = -0.2\}$ (b) $\{z_1 = 0.2, z_2 = -0.2 - 0.1i\}$ (c) $\{z_1 = 0.2e^{i\frac{\pi}{4}}, z_2 = -0.2e^{-i\frac{\pi}{4}}\}$

G. Zhu, O. Kálmán, K. Wang, L. Xiao, X. Zhan, Z. Bian, T. Kiss, P. Xue, Phys. Rev. A 100, 052307 (2019)

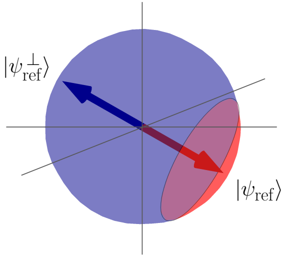
Quantum state matching



- define a reference state: $|\psi_{\mathrm{ref}}\rangle$
- define a neighborhood: $\varepsilon = |\langle\psi \mid \psi_{\mathrm{ref}}\rangle|$
- find which f corresponds to it
- find implementation of f
  - 2-qubit unitary+post-selection

O. Kálmán and T. Kiss, Phys. Rev. A **97**, 032125 (2018)

Quantum state matching



- define a reference state: $|\psi_{\text{ref}}\rangle$
- define a neighborhood: $\varepsilon = |\langle\psi \mid \psi_{\text{ref}}\rangle|$
- find which f corresponds to it
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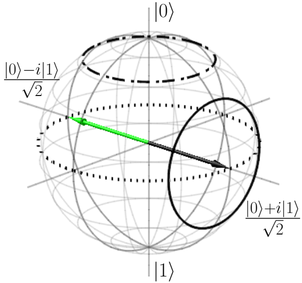
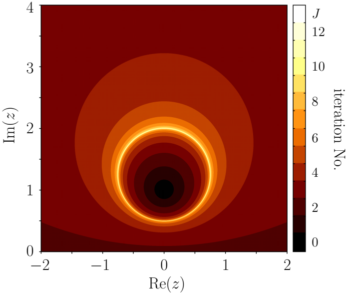
  - 2-qubit unitary+post-selection





$$|\psi_{\text{ref}}\rangle = \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}}$$

$$\varepsilon^2 = 0.9$$

O. Kálmán and T. Kiss, Phys. Rev. A **97**, 032125 (2018)

Perspectives

- Photonic experiment: two iterations of an ergodic map
- Noisy evolution?
  Competition between decoherence and purification: quaternionic representation and quaternionic fractals
  by David Viennot
  https://arxiv.org/abs/2003.02608
- Implementation on real quantum computers
  (e.g. IBM Q, Google, etc.)

Thank you for your attention!

MAGYAR TUDOMÁNYOS AKADÉMIA 1825

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Quantum Information National Laboratory