Activation of metrologically useful genuine multipartite entanglement

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   - Quantum metrology
2. Improving metrological performance
   - Taking many copies
   - Embedding into higher dimension

Basic task in quantum metrology

Linear interferometer — Quantum measurement

$$\varrho \Rightarrow \boxed{U_\theta = \exp(-i\mathcal{H}\theta)} \Rightarrow U_\theta \varrho U_\theta^\dagger \Rightarrow \boxed{\text{Estimation of } \theta}$$

- $\mathcal{H}$ is *local*, that is,

$$\mathcal{H} = h_1 + \cdots + h_N,$$

where h_n's are single-subsystem operators of the N-partite system.

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where h_n's are single-subsystem operators of the N-partite system.

- Cramér-Rao bound:

$$(\Delta\theta)^2 \geq \frac{1}{\mathcal{F}_Q[\varrho, \mathcal{H}]},$$

where the quantum Fisher information is

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|\mathcal{H}|l\rangle|^2,$$

with $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ being the eigendecomposition.

Scaling properties of the quantum Fisher information

General derivations yield: [G. Tóth and I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- The maximum for separable states (shot-noise scaling)

  [L. Pezzé and A. Smerzi, PRL 102, 100401 (2009)] [P. Hyllus et al., PRA 82, 012337 (2010)]

  $$\mathcal{F}_Q[\varrho, \mathcal{H}] \sim N \quad \xrightarrow{\text{Cramér-Rao}} \quad (\Delta\theta)^2 \sim 1/N$$

- The maximum for k-entangled states

  [P. Hyllus et al., PRA 85, 022321 (2012)] [G. Tóth, PRA 85, 022322 (2012)]

  $$\mathcal{F}_Q[\varrho, \mathcal{H}] \sim kN \quad \xrightarrow{\text{Cramér-Rao}} \quad (\Delta\theta)^2 \sim 1/kN$$

- The maximum for (genuine multipartite) entangled states (Heisenberg scaling)

  $$\mathcal{F}_Q[\varrho, \mathcal{H}] \sim N^2 \quad \xrightarrow{\text{Cramér-Rao}} \quad (\Delta\theta)^2 \sim 1/N^2$$

The metrological gain for characterizing usefulness

- For a given ϱ and a *local* Hamiltonian $\mathcal{H} = h_1 + \cdots + h_N$

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})},$$

← Performance of ϱ with $\mathcal{H}$

← Best performance of all *separable* states with $\mathcal{H}$

where the separable limit is

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1}^{N} [\sigma_{\max}(h_n) - \sigma_{\min}(h_n)]^2.$$

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If $\sigma_{\mathrm{max/min}}(h_n) = \pm 1 \rightarrow$
- $\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = 4N$
- $\max \mathcal{F}_Q[\varrho, \mathcal{H}] = 4N^2$ for some entangled ϱ with a local $\mathcal{H}$.

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- $g_{\mathcal{H}}(\varrho)$ can be maximized over *local* Hamiltonians [G. Tóth et al., PRL 125, 020402 (2020)]

$$g(\varrho) = \max_{\mathrm{local}\mathcal{H}} g_{\mathcal{H}}(\varrho).$$

- If $g(\varrho) > 1$ then the state is useful metrologically.

The metrological gain witnesses multipartite entanglement

- Fully-separable states $\rightarrow g \leq 1$ (shot-noise scaling).
- Entanglement is required for usefulness but not all entangled states are useful.
- PPT entangled states can be useful. [G. Tóth and T. Vértesi, PRL 120, 020506 (2018)]

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- g identifies different levels of multipartite entanglement.
- $g > k \rightarrow$ *metrologically useful* $(k+1)$-partite entanglement.
- $g > N-1 \rightarrow$ *metrologically useful* N-partite/genuine multipartite entanglement (GME).
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- $g = N$ ($\mathcal{F}_Q = 4N^2$) is the maximal usefulness (Heisenberg scaling).
- There are non-useful GME states [P. Hyllus et al., PRA 82, 012337 (2010)]
- What kind of entangled states can be made useful with extended techniques?

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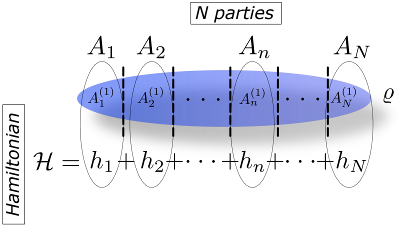
Multicopy scheme with interaction between the copies

The single-subsystem operators h_n's act between the copies:



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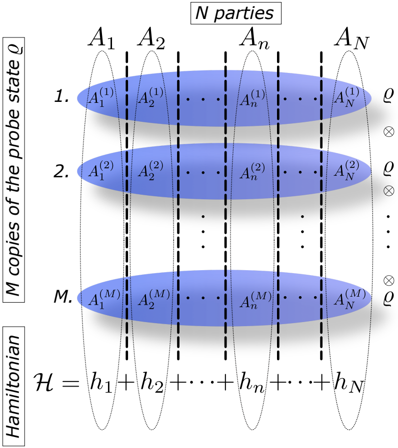

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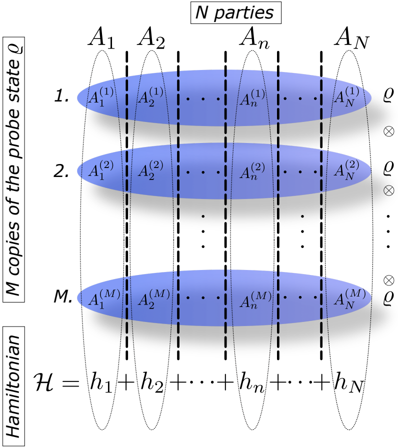

The gain can be improved $g(\varrho^{\otimes M}) > g(\varrho)$! [G. Tóth et al., PRL 125, 020402 (2020)]

Metrologically useful GME activation

Result

Entangled states of $N \geq 2$ qudits of dimension d are maximally useful in the infinite copy limit if they live in the subspace

$$\{|0..0\rangle, |1..1\rangle, ..., |d-1, .., d-1\rangle\}.$$

The maximum is attained exponentially fast with the number of copies.

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$$\varrho = \sum_{k,l=0}^{d-1} c_{kl}(|k\rangle\langle l|)^{\otimes N}$$

$$h_n = D^{\otimes M}, \text{ for } 1 \leq n \leq N$$

$$D = \mathrm{diag}(+1, -1, +1, -1, ...)$$

$$\text{for qubits} \rightarrow D = \sigma_z \text{, and } h_n = \sigma_z^{\otimes M}$$

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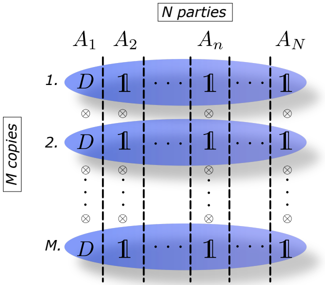

$$\mathcal{H} = h_1 + h_2 + \cdots + h_n + \cdots + h_N$$

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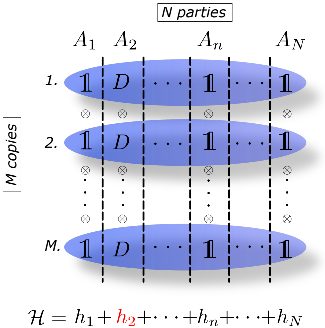

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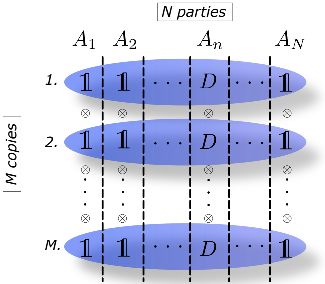

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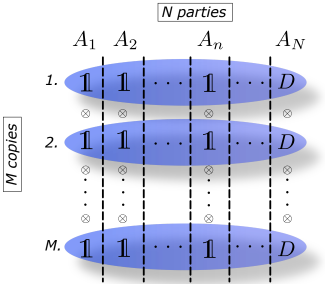

$$\mathcal{H} = h_1 + h_2 + \cdots + h_n + \cdots + h_N$$

Examples

- The state with $|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$

$$\varrho_N(p) = p\,|\mathrm{GHZ}_N\rangle\langle\mathrm{GHZ}_N| + (1-p)\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2}.$$

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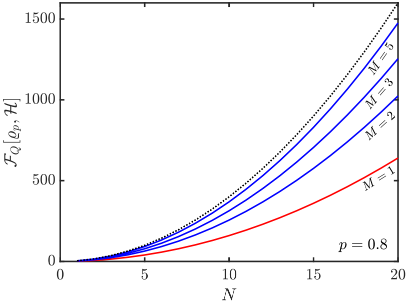

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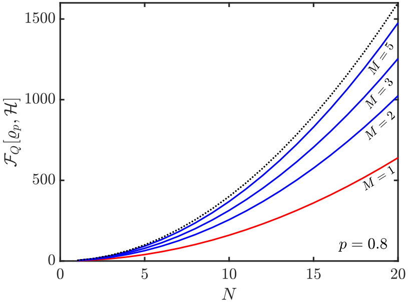

- All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k\, |k\rangle^{\otimes N}.$$

Phase noise for $N = 3$, $M = 1$ copy

$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ with $\mathcal{H} = h_1 + h_2 + h_3$, where $h_n = \sigma_z$ so $\mathcal{H} = \sigma_z^{(1)} + \sigma_z^{(2)} + \sigma_z^{(3)}$.

For $M = 1$ copy:

$$\begin{aligned} \mathcal{F}_Q[|\mathrm{GHZ}\rangle, \mathcal{H}] &= 36 = 4N^2 \,(\mathrm{maximal}), \\ \mathcal{F}_Q[\varrho, \mathcal{H}] &< 36, \end{aligned}$$

with

$$\varrho = p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\,|\mathrm{GHZ}_\phi\rangle\langle\mathrm{GHZ}_\phi|,$$

where $|\mathrm{GHZ}_\phi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + e^{-i\phi}|111\rangle)$.

- So ϱ is a mixture of $|\mathrm{GHZ}\rangle$ and the phase-error affected $|\mathrm{GHZ}\rangle$.
- For 1 copy, the quantum Fisher information decreases if there is a phase-error.

Tolerating phase noise for $N=3$, $M=3$ copies

$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ with $\mathcal{H} = h_1 + h_2 + h_3$, where $h_n = \sigma_z^{\otimes M}$.

For $M=3$ copies:

$$\begin{aligned} \mathcal{F}_Q[|\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle, \mathcal{H}] &= 36 = 4N^2 \,(\mathrm{maximal}), \\ \mathcal{F}_Q[\varrho, \mathcal{H}] &= 36, \end{aligned}$$

where ϱ is some mixture of states with phase-error on at most 1 copy:

$$\begin{aligned} &|\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle, \\ &|\mathrm{GHZ}_{\phi_1}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle, \\ &|\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}_{\phi_2}\rangle \otimes |\mathrm{GHZ}\rangle, \\ &|\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}_{\phi_3}\rangle. \end{aligned}$$

- For 3 copies, the quantum Fisher information stays maximal if there is a phase-error on at most 1 copy.
- Adding more copies protects against phase-error on 1 copy.

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Embedding "GHZ"-like states can make them useful

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All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k \left|k\right\rangle^{\otimes N}$$

with $\sum_k |\sigma_k|^2 = 1$ are useful for $d \geq 3$ and $N \geq 3$.

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- The state for $N \geq 3$ with $d = 2$

$$|\psi\rangle = \sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N}$$

is useful if $1/N < 4|\sigma_0\sigma_1|^2$ [P. Hyllus et al., PRA 82, 012337 (2010)].

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  is useful if $1/N < 4|\sigma_0\sigma_1|^2$ [P. Hyllus et al., PRA 82, 012337 (2010)].
- But with $d = 3$

$$|\psi'\rangle = \sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N} + 0 |2\rangle^{\otimes N}$$

  is always useful.
- The non-useful $|\psi\rangle$, embedded into $d = 3$ ($|\psi'\rangle$) becomes useful.

Conclusions

- Investigated the metrological performance of quantum states in the multicopy scenario.
- Identified a subspace in which metrologically useful GME activation is possible.
- Also improved metrological performance by embedding.

See New J. Phys. 26 023034 (2024)!

Thank you for the attention!

QUANTERA
Alexander von Humboldt Stiftung/Foundation
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EUSKO JAURLARITZA GOBIERNO VASCO
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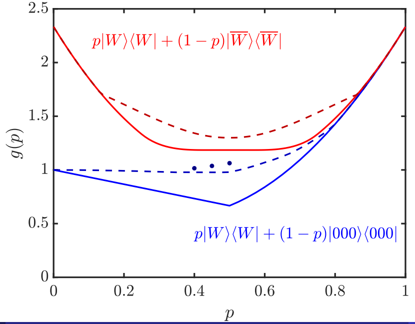
States outside the previous subspace

- For $N = 3$ with the states

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$
$$|\overline{W}\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle)$$

- Using the numerical optimization for $g(\varrho)$ [G. Tóth et al., PRL 125, 020402 (2020)].



Optimal measurements

- In the limit of many copies ($M \gg 1$)

$$\mathcal{F}_Q[\varrho_N(p)^{\otimes M}, \mathcal{H}] = 4N^2 \implies (\Delta\theta)^2 \geq 1/\mathcal{F}_Q[\varrho_N(p)^{\otimes M}, \mathcal{H}] = 1/4N^2$$

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- Measuring in the eigenbasis of $\mathcal{M}$ (error propagation formula):

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{(\Delta\mathcal{M})^2}{|\partial_\theta \langle\mathcal{M}\rangle|^2} = \frac{(\Delta\mathcal{M})^2}{\langle i[\mathcal{M}, \mathcal{H}]\rangle^2}.$$

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- For M copies of $\varrho_N(p)$ we constructed a simple $\mathcal{M}$ such that

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{1 + (M-1)p^2}{4MN^2p^2}$$

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- For $M = 2$ copies of $\varrho_3(p)$

$$\mathcal{M} = \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} + \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y$$

The general measurements for Observation 1

with
$$\varrho(p,q,r) = p\,|\mathrm{GHZ}_q\rangle\langle\mathrm{GHZ}_q| + (1-p)[r(|0\rangle\langle 0|)^{\otimes N} + (1-r)(|1\rangle\langle 1|)^{\otimes N}],$$
$$|\mathrm{GHZ}_q\rangle = \sqrt{q}\,|000..00\rangle + \sqrt{1-q}\,|111..11\rangle\,,$$
The following operator, being the sum of M correlation terms
$$\mathcal{M} = \sum_{m=1}^{M} Z^{\otimes(m-1)} \otimes Y \otimes Z^{\otimes(M-m)},$$
where we define the operators acting on a single copy
$$Y = \begin{cases} \sigma_y^{\otimes N} & \text{for odd } N, \\ \sigma_x \otimes \sigma_y^{\otimes(N-1)} & \text{for even } N, \end{cases}$$
$$Z = \sigma_z \otimes \mathbb{1}^{\otimes(N-1)}.$$
$$(\Delta\theta)^2_{\mathcal{M}} = \frac{1/[4q(1-q)] + (M-1)p^2}{4MN^2p^2}.$$

Observation

Full-rank states of N qudits cannot be maximally useful in the infinite copy limit.

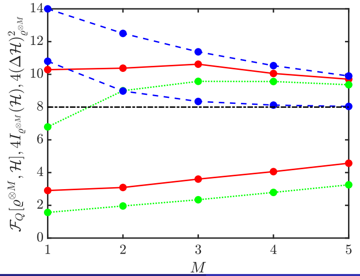
- *Example*: Isotropic state of two qubits

$$\varrho^{(p)} = p\,|\Psi_{\rm me}\rangle\langle\Psi_{\rm me}| + (1-p)\mathbb{1}/2^2,$$

where $|\Psi_{\rm me}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

- $\varrho^{(0.75)}$ (top 3 curves) and $\varrho^{(0.35)}$ (bottom 3 curves). $h_n = \sigma_z^{\otimes M}$.

$$4(\Delta\mathcal{H})^2 \geq \mathcal{F}_Q[\varrho,\mathcal{H}] \geq 4I_\varrho(\mathcal{H})$$



Embedding mixed states

- Embedding the noisy GHZ state

$$\varrho_N^{(p)} = p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{\mathbb{1}}{2^N}.$$

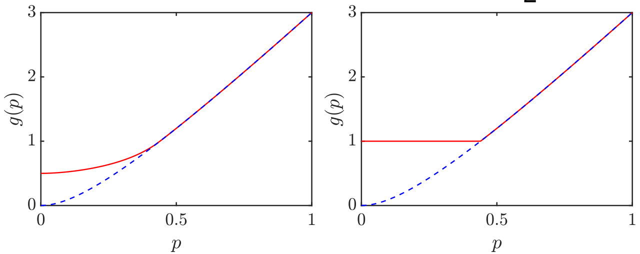


Figure: The metrological gain for the state $\varrho_3^{(p)}$ (dashed), embedded into $d=3$ (left), $d=4$ (right).

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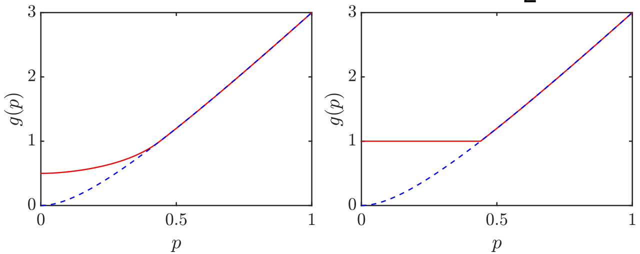

Figure: The metrological gain for the state $\varrho_3^{(p)}$ (dashed), embedded into $d = 3$ (left), $d = 4$ (right).

- $\varrho_3^{(p)}$ is genuine multipartite entangled for $p > 0.428571$ [SM Hashemi Rafsanjani et al., PRA 86, 062303 (2012)].
- $\varrho_3^{(p)}$ is useful metrologically for $p > 0.439576$.

Error propagation formula

- Measuring in the eigenbasis of $\mathcal{M}$ we get:

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{(\Delta\mathcal{M})^2}{|\partial_\theta\langle\mathcal{M}\rangle|^2} = \frac{(\Delta\mathcal{M})^2}{\langle i[\mathcal{M},\mathcal{H}]\rangle^2}.$$

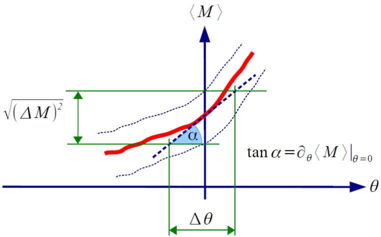

Figure from [G. Tóth and I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)].

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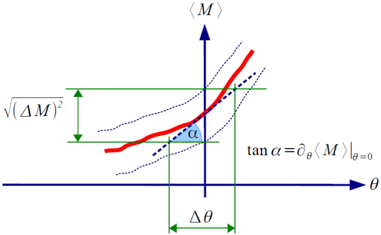


Figure from [G. Tóth and I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)].

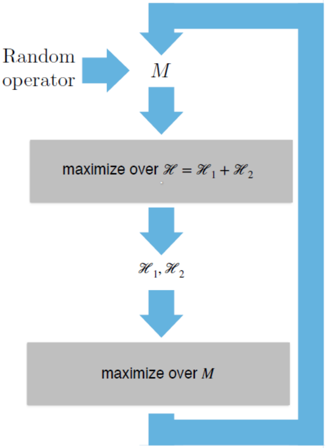
- From the Cramér-Rao bound it follows that for any $\mathcal{M}$

$$\frac{(\Delta\mathcal{M})^2}{\langle i[\mathcal{M},\mathcal{H}]\rangle^2} = (\Delta\theta)^2_{\mathcal{M}} \geq \frac{1}{\mathcal{F}_Q[\varrho,\mathcal{H}]}$$

See-saw method for optimizing the gain

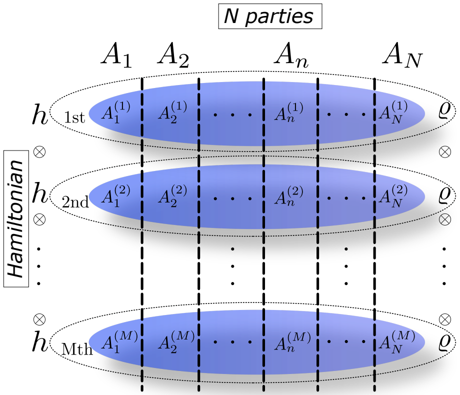
- Used in [G. Tóth et al., PRL 125, 020402 (2020)].
- Minimizing $(\Delta\theta)^2_{\mathcal{M}} = \frac{(\Delta\mathcal{M})^2}{\langle i[\mathcal{M},\mathcal{H}]\rangle^2} \geq \frac{1}{\mathcal{F}_Q[\varrho,\mathcal{H}]}$ with constraints $c_n\mathbf{1} \pm h_n \geq 0$.
- For given ϱ and $\mathcal{H} = h_1 + h_2$ the symmetric logarithmic derivate gives the optimum

$$\mathcal{M}_{opt} = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| \langle k|\mathcal{H}|l\rangle$$



Scheme without interaction between copies

Consider M copies of an N-partite state ϱ, all undergoing a dynamics governed by the same Hamiltonian h:



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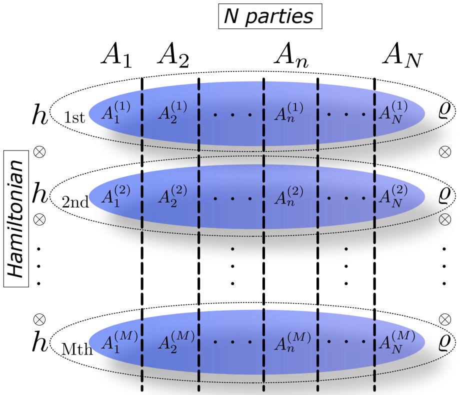

$$\mathcal{F}_Q[\varrho^{\otimes M}, h^{\otimes M}] = M\mathcal{F}_Q[\varrho, h],$$

but the separable maximum also increases

$$\mathcal{F}_Q^{(\text{sep})}(h^{\otimes M}) = M\mathcal{F}_Q^{(\text{sep})}(h).$$

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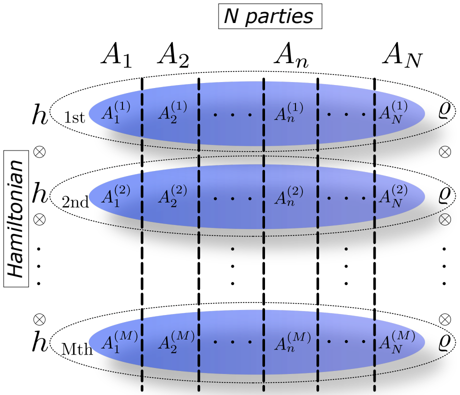

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No improvement in the gain!

An example for $N = 3$

Consider the state

$$\varrho_3(p) = p\,|\mathrm{GHZ}_3\rangle\langle\mathrm{GHZ}_3| + \frac{1-p}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right),$$

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$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}_{M=1}) = \mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}_{M=2}) = 12.$$