Strong converses and Rényi divergences in Quantum Information Theory

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Joint work with: Tomohiro Ogawa, Mark Wilde, Tom Cooney

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arXiv:1408.3373, arXiv:1408.6894, arXiv:1409.3562

Rényi divergences

- p, q probability distributions on $\mathcal{X}$, $\quad \alpha \in [0, +\infty) \setminus \{1\}$:

$$D_\alpha\left(p \| q\right) := \frac{1}{\alpha - 1} \log \sum\nolimits_x p(x)^\alpha q(x)^{1-\alpha}$$

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- Approximation of the relative entropy (Kullback-Leibler divergence):

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- Nice mathematical properties: positivity, monotonicity under stochastic maps, etc.

- Operational significance: Quantifies the trade-off between the relevant quantities in many coding problems.

Trade-off relations

- Most coding problems are characterized by two competing quantities, an error and a rate.

	error	rate
channel coding	decoding error	coding rate
state compression	decompression error	compression rate
binary hypothesis testing	type I error	type II error rate

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  Strong converse property:

  For rates above the optimal, the error goes to 1.

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- Direct exponent: The optimal exponential decay of the error probability when the rate is below the optimal.

  Strong converse exponent: The optimal exponential decay of the success probability when the rate is above the optimal.

Classical trade-off relations

- X_α: relevant Rényi quantity of the problem

	X_α
channel coding	Rényi capacity
state compression	Rényi entropy
binary hypothesis testing	Rényi divergence

- Direct rate:

$$d(r) = \sup_{0<\alpha<1} \frac{\alpha - 1}{\alpha} [r - X_\alpha]$$

- Strong converse rate:

$$sc(r) = \sup_{1<\alpha} \frac{\alpha - 1}{\alpha} [r - X_\alpha]$$

- Operational interpretation of X_α.

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$$D_\alpha^{(\text{new})}(\rho\,\|\,\sigma) := \frac{1}{\alpha - 1} \log \operatorname{Tr} \left(\rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{\alpha}} \rho^{\frac{1}{2}}\right)^\alpha$$

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- Are either of these the right quantum generalizations?

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- $D_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log \operatorname{Tr} \rho^\alpha \sigma^{1-\alpha}$ or $\frac{1}{\alpha-1}\log \operatorname{Tr}\left(\rho^{\frac{1}{2}}\sigma^{\frac{1-\alpha}{\alpha}}\rho^{\frac{1}{2}}\right)^\alpha$?

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  Direct and strong converse rates in quantum hypothesis testing and other problems.

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- Other versions may also be of interest.

Mathematical properties

- Both $D_\alpha^{(\mathrm{old})}$ and $D_\alpha^{(\mathrm{new})}$ are monotone increasing in α

$$\lim_{\alpha\to 1} D_\alpha^{(x)}(\rho\|\sigma) = D_1(\rho\|\sigma) := D(\rho\|\sigma) := \mathrm{Tr}\,\rho(\log\rho - \log\sigma)$$

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- Araki-Lieb-Thirring inequality:

$$D_\alpha^{(\text{old})}(\rho\,\|\,\sigma) \geq D_\alpha^{(\text{new})}(\rho\,\|\,\sigma)\,, \qquad \alpha \in [0, +\infty]$$

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- Audenaert's converse ALT:

$$D_\alpha^{(\mathrm{new})}(\rho\,\|\,\sigma) \geq \alpha D_\alpha^{(\mathrm{old})}(\rho\,\|\,\sigma) - |\alpha-1|\log\dim\mathcal{H}$$

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- Monotonicity: Φ CPTP

$$D_\alpha^{(\mathrm{old})}\left(\Phi(\rho)\,\|\,\Phi(\sigma)\right) \le D_\alpha^{(\mathrm{old})}\left(\rho\,\|\,\sigma\right), \qquad \alpha \in [0,2]$$

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- Sufficiency for $D_\alpha^{(\mathrm{old})}$, $\alpha \in (0,2)$:

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$$\Longleftrightarrow \exists\Psi : \quad \Psi(\Phi(\rho)) = \rho, \qquad \Psi(\Phi(\sigma)) = \sigma$$

Petz recovery map

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  Petz recovery map

- $D_{1/2}^{(\mathrm{new})}$ and $D_\infty^{(\mathrm{new})}$ don't satisfy sufficiency

  (because they can be achieved by measurements)

  How about $\alpha \in (1/2,+\infty)$?

Operational interpretation?

Quantum hypothesis testing

- Two candidates for the true state of a system: $H_0 : \rho$ vs. $H_1 : \sigma$

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- Quantum Stein's lemma:[1]

$$\alpha_n(T_n) \to 0 \implies \beta_n(T_n) \sim e^{-nD_1(\rho\|\sigma)}$$ is the optimal decay

[1] Hiai, Petz, 1991, Ogawa, Nagaoka, 2001.

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Operational interpretation of the relative entropy.

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Quantifying the trade-off

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- Direct domain: Quantum Hoeffding bound[1]

$$\beta_n(T_n) \sim e^{-nr} \implies \alpha_n(T_n) \sim e^{-nH_r}, \qquad r < D_1(\rho\|\sigma)$$

Converse domain: Quantum Han-Kobayashi bound[2]

$$\beta_n(T_n) \sim e^{-nr} \implies \alpha_n(T_n) \sim 1 - e^{-nH_r^*}, \qquad r > D_1(\rho\|\sigma)$$

- Hoeffding divergence/anti-divergence:

$$H_r := \sup_{0<\alpha<1} \frac{\alpha-1}{\alpha}\left[r - D_\alpha^{(\mathrm{old})}(\rho\,\|\,\sigma)\right]$$

$$H_r^* := \sup_{1<\alpha} \frac{\alpha-1}{\alpha}\left[r - D_\alpha^{(\mathrm{new})}(\rho\,\|\,\sigma)\right]$$

[1] Hayashi 2006; Nagaoka 2006; Audenaert, Nussbaum, Szkoła, Verstraete 2007
[2] Ogawa, Nagaoka 2000; Hayashi 2006; Mosonyi, Ogawa, 2013

Moral

The right quantum extension is $D_\alpha^{(\mathrm{old})}$ for $\alpha < 1$ and $D_\alpha^{(\mathrm{new})}$ for $\alpha > 1$.

$$D_\alpha(\rho\|\sigma) := \begin{cases} D_\alpha^{(\mathrm{old})}(\rho\|\sigma), & \alpha \in [0,1), \\ D_\alpha^{(\mathrm{new})}(\rho\|\sigma), & \alpha \in (1,+\infty]. \end{cases}$$

Monotonicity holds for every $\alpha \in [0,+\infty]$.

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Monotonicity holds for every $\alpha \in [0,+\infty]$.

Remark: $D_{1/2}^{(\mathrm{new})}(\rho\|\sigma) = -2\log F(\rho\|\sigma)$ fidelity

Doesn't seem to have an operational interpretation.

Moral

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Trade-off relations for other coding theorems?

Correlated states

Hypothesis testing for correlated states

- Gibbs states on a spin chain,
- temperature states of non-interacting fermions/bosons on a lattice (gauge-invariant Gaussian states)

The trade-off relations are quantified by the regularized Rényi divergences[1]

$$D_\alpha^{(\mathrm{x})}(\rho\|\sigma) := \lim_{n\to+\infty} \frac{1}{n} D_\alpha^{(\mathrm{x})}(\rho_n\|\sigma_n)$$

$$\mathrm{x} = \begin{cases} \mathrm{old}, & \text{for the direct domain}, \quad 0 < \alpha < 1, \\ \mathrm{new}, & \text{for the converse domain}, \quad 1 < \alpha. \end{cases}$$

[1] Hiai, Fannes, Mosonyi, Ogawa 2008; Mosonyi, Ogawa 2014

Mutual information and conditional entropy

- H_0: $\rho_{AB}^{\otimes n}$ vs. H_1 : $\tau_A^{\otimes n} \otimes \mathcal{S}(\mathcal{H}^{\otimes n})$
- Relevant Rényi quantities:

$$D_\alpha^{(\mathrm{x})}(\rho_{AB} \| \tau_A) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} D_\alpha^{(\mathrm{x})}(\rho_{AB} \| \tau_A \otimes \sigma)$$

- Direct rate:[1]

$$d(r) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} H_r(\rho_{AB} \| \tau_A \otimes \sigma) = \sup_{0<\alpha<1} \frac{\alpha - 1}{\alpha} \left[r - D_\alpha^{(\mathrm{old})}(\rho_{AB} \| \tau_A) \right]$$

- Strong converse rate:[1]

$$sc(r) = \sup_{\sigma \in \mathcal{S}(\mathcal{H})} H_r^*(\rho_{AB} \| \tau_A \otimes \sigma) = \sup_{0<\alpha<1} \frac{\alpha - 1}{\alpha} \left[r - D_\alpha^{(\mathrm{new})}(\rho_{AB} \| \tau_A) \right]$$

- $\tau_A = \rho_A$: mutual information; $\tau_A = I_A / d_A$: conditional entropy.

[1] Hayashi, Tomamichel, 2014

Channel discrimination

- null-hypothesis: $\mathcal{N}_1$; alternative hypothesis: $\mathcal{N}_2$

  CPTP maps from system A to system B.

- Non-adaptive strategy: Feed in $\psi_{R_n A^n}$, measure the outcome.

  Product strategy: $\psi_{R_n A^n} = \psi_{RA}^{\otimes n}$

- channel divergences:

$$D_\alpha^{(\mathrm{x})}(\mathcal{N}_1 \| \mathcal{N}_2) := \sup_{\psi_{RA}} D_\alpha^{(\mathrm{x})}(\mathcal{N}_1 \psi_{RA} \| \mathcal{N}_2 \psi_{RA})$$

  Hoeffding divergence $H_r(\mathcal{N}_1 \| \mathcal{N}_2)$ and anti-divergence $H_r^*(\mathcal{N}_1 \| \mathcal{N}_2)$ defined as before

- Only product strategies allowed: The trade-off relations are quantified by the channel divergences.

Channel discrimination

- null-hypothesis: $\mathcal{N}_1$; alternative hypothesis: $\mathcal{N}_2$
- adaptive discrimination strategy is allowed
- classical channels: (Hayashi 2009)

  trade-off relations are given by $H_r(\mathcal{N}_1\|\mathcal{N}_2)$ and $H_r^*(\mathcal{N}_1\|\mathcal{N}_2)$

  No advantage from adaptive strategies.
- $\mathcal{N}_2(.) = \mathrm{Tr}(.)\sigma$ replacer channel: (Cooney, Mosonyi, Wilde 2014)

  strong converse exponent is given by $H_r^*(\mathcal{N}_1\|\mathcal{N}_2)$

  No advantage from adaptive strategies.
- both channels are replacers: $\mathcal{N}_1(.) = \mathrm{Tr}(.)\rho$, $\mathcal{N}_2(.) = \mathrm{Tr}(.)\sigma$

  state discrimination

Classical-quantum channels

- Channel: $W : \mathcal{X} \to \mathcal{S}(\mathcal{H})$
- memoryless extensions: classical-quantum channel

$$W^{\otimes n}(x_1, \ldots, x_n) := W(x_1) \otimes \ldots \otimes W(x_n), \qquad x_i \in \mathcal{X}$$

- Code: $\mathcal{C}^{(n)} = (\mathcal{C}_e^{(n)}, \mathcal{C}_d^{(n)})$

$\mathcal{C}_e^{(n)} : \{1, \ldots, M_n\} \to \mathcal{X}^n$ encoding

$\mathcal{C}_d^{(n)} : \{1, \ldots, M_n\} \to \mathcal{B}(\mathcal{H}^{\otimes n})_+$ decoding POVM

- average error probability:

$$p_e\left(W^{\otimes n}, \mathcal{C}^{(n)}\right) := \frac{1}{M_n} \sum_{k=1}^{M_n} \operatorname{Tr} W^{\otimes n}(\mathcal{C}_e^{(n)}(k))(I - \mathcal{C}_d^{(n)}(k))$$

- classical capacity:

$$C(W) := \max\left\{\liminf_{n\to+\infty} \frac{1}{n} \log M_n : p_e\left(W^{\otimes n}, \mathcal{C}^{(n)}\right) \to 0\right\}$$

Classical-quantum channels

- **Channel:** $W : \mathcal{X} \to \mathcal{S}(\mathcal{H}) \qquad \widehat{W} : x \mapsto |x\rangle\langle x| \otimes W_x$

$$\widehat{W}(p) := \sum_{x \in \mathcal{X}} p(x)|x\rangle\langle x| \otimes W(x) \qquad \text{classical-quantum state}$$

- **α-Holevo quantities:**

$$\chi_\alpha^{(x)}(W,p) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} D_\alpha^{(x)}(\widehat{W}(p) \| p \otimes \sigma)$$

$$\lim_{\alpha \to 1} \chi_\alpha^{(x)}(W,p) = \chi(W,p) := S\left(W(p)\right) - \sum_{x \in \mathcal{X}} p(x) S(W(x))$$

- **Theorem:**[1]

$$C(W) = \sup_p \chi(W,p)$$

[1] Holevo, 1996; Schumacher-Westmoreland, 1997

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- **Theorem:**[1]

$$sc(r) = \sup_{1<\alpha} \frac{\alpha - 1}{\alpha} \left[r - \sup_p \chi_\alpha^{(\text{new})}(W,p) \right]$$

- Direct rate is unknown even classically.

[1] Mosonyi, Ogawa 2014

- Yet another quantum Rényi divergence:

$$D_\alpha^\flat(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \operatorname{Tr} e^{\alpha \log \rho + (1-\alpha) \log \sigma}$$

- Theorem:[1]

$$D_\alpha^\flat(\rho\|\sigma) = \sup_{\tau \in \mathcal{S}_\rho(\mathcal{H})} \left\{ D(\tau\|\sigma) - \frac{\alpha}{\alpha - 1} D(\tau\|\rho) \right\}.$$

[1] Hiai, Petz, 1993; Mosonyi, Ogawa, 2014

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- Not monotone.

$$D_\alpha^\flat(\rho\|\sigma) \le D_\alpha^{(\mathrm{new})}(\rho\|\sigma) \le D_\alpha^{(\mathrm{old})}(\rho\|\sigma)$$

[1] Hiai, Petz, 1993; Mosonyi, Ogawa, 2014

Summary

	$d(r)$	$sc(r)$
binary state discrimination	$D_\alpha^{(\mathrm{old})}(\rho\|\sigma)$	$D_\alpha^{(\mathrm{new})}(\rho\|\sigma)$
binary state discrimination bipartite, composite H_1	$D_\alpha^{(\mathrm{old})}(\rho_{AB}\|\tau_A)$	$D_\alpha^{(\mathrm{new})}(\rho_{AB}\|\tau_A)$
binary channel discrimination adaptive, H_1 replacer	?	$D_\alpha^{(\mathrm{new})}(\mathcal{N}_1\|\mathcal{N}_2)$
binary channel discrimination adaptive	?	?
classical-quantum channel coding	?	$\sup_p \chi_\alpha^{(\mathrm{new})}(W,p)$

Other problems? Quantum capacity, entanglement-assisted capacity, etc.?

Conditional Rényi mutual information?