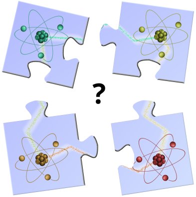
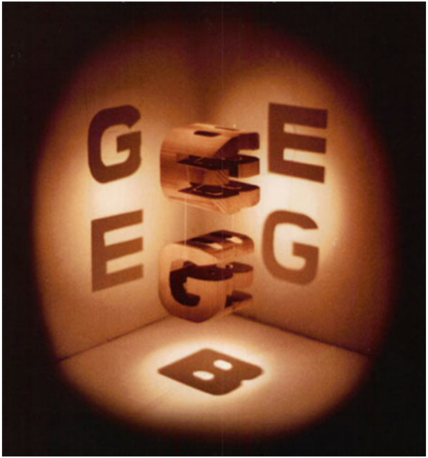
The Quantum Marginal Problem

Otfried Gühne
F. Huber, H. C. Nguyen, J. Siewert, T. Simnacher, N. Wyderka, X.-D. Yu,

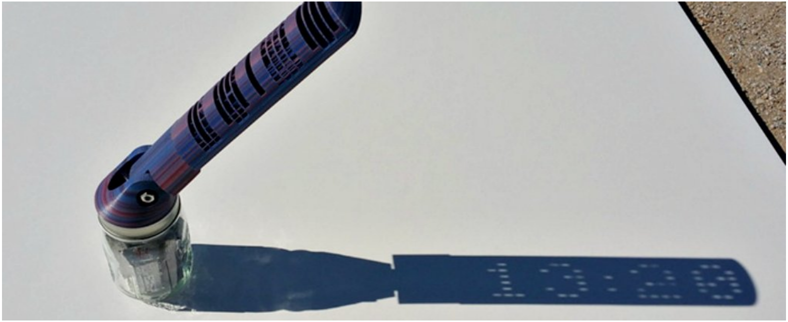


Department Physik, Universität Siegen

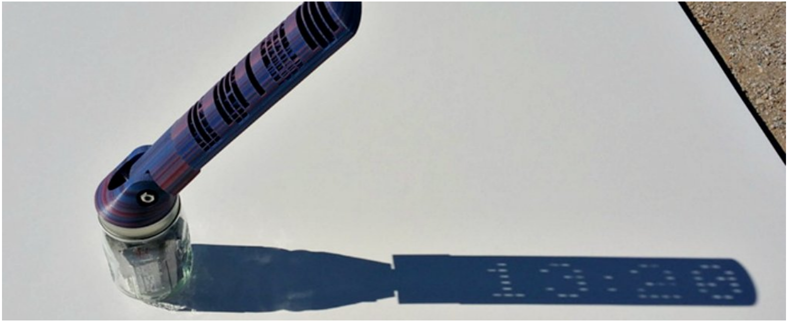
Gödel, Escher, Bach



Digital sundial



Digital sundial



Theorem (Falconer, 1987)

Consider 2D shadows in all spatial directions. Then there is a 3D object having these shadows (up to measure zero).

Marginal distributions

Question

Can $p(x, y, z)$ be reconstructed from $p(x, y)$, $p(y, z)$, and $p(x, z)$?

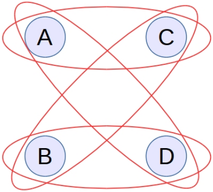
Marginal distributions

Question

Can $p(x, y, z)$ be reconstructed from $p(x, y)$, $p(y, z)$, and $p(x, z)$?

Example

Consider 4 variables A, B, C, D with values ± 1 and the marginal distributions (A, C), (A, D), (B, C) and (B, D). When do they come from a global distribution?



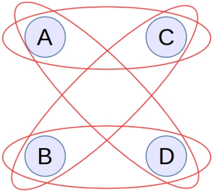
Marginal distributions

Question

Can $p(x, y, z)$ be reconstructed from $p(x, y)$, $p(y, z)$, and $p(x, z)$?

Example

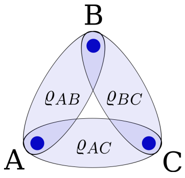
Consider 4 variables A, B, C, D with values ± 1 and the marginal distributions (A, C), (A, D), (B, C) and (B, D). When do they come from a global distribution?

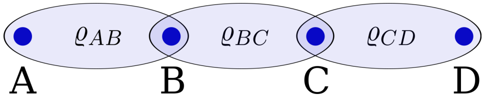


Iff they obey the CHSH inequality, A. Fine, PRL 48, 291 (1981).

The quantum case

- How do local properties determine the global properties of a quantum state?
- Which quantum states are determined as thermal states of a local Hamiltonian?





Maximally entangled states

How entangled can two couples get?

A. Higuchi, A. Sudbery *

Dept. of Mathematics, University of York, Heslington, York, YO10 5DD, UK

Results and Questions

- A bipartite pure state is maximally entangled, if the marginals are maximally mixed.
- For four qubits, there is no state that is maximally entangled for any bipartition.
- What happens for general states of N particles?

Phys. Lett. A 273, 213 (2000)

Three qubits

VOLUME 89, NUMBER 20 PHYSICAL REVIEW LETTERS 11 NOVEMBER 2002

Almost Every Pure State of Three Qubits Is Completely Determined by Its Two-Particle Reduced Density Matrices

N. Linden,[1] S. Popescu,[2] and W. K. Wootters[3]
[1]*School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW United Kingdom*

Results and Questions

- Nearly all pure three-qubit states are determined by their reduced two-body marginals.
- $\Rightarrow$ All pure three-qubit states can be approximated by ground states of two-body Hamiltonians.
- For more qubits, are there states which cannot be approximated by two-body thermal states?

Graph states

PHYSICAL REVIEW A **77**, 012301 (2008)

Graph states as ground states of many-body spin-1/2 Hamiltonians

M. Van den Nest,[1] K. Luttmer,[1] W. Dür,[1,2] and H. J. Briegel[1,2]

Results and Questions

- Graph states cannot be exact ground states of two-body Hamiltonians.
- If they can be approximated, then the energy gap vanishes.
- But can one approximate them at all? Or is there a finite distance?

Outline

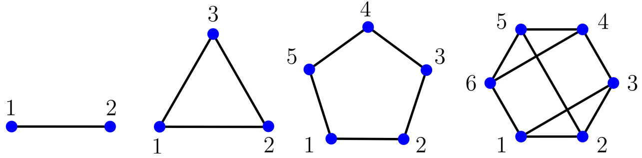
Questions

- Given a set of reduced states, is there a global state compatible with it?
- Given a global state, is it uniquely determined by its reduced states?
- Given a global state, which properties can be inferred by looking at the marginals only?

Outline

1. Are there N-particle pure states, for which many marginals are maximally mixed?
2. How can we address the general pure state marginal problem?

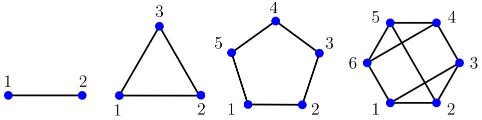
Maximally entangled states



Absolutely maximally entangled states

Results on AME states

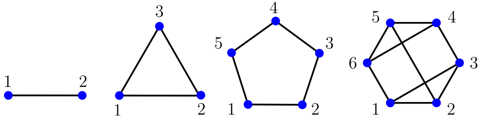
- An N-particle state where all $\lfloor N/2 \rfloor$-particle reduced states are maximally mixed is called AME.
- Examples: Bell states, GHZ states, quantum codewords, ...



Absolutely maximally entangled states

Results on AME states

- An N-particle state where all $\lfloor N/2 \rfloor$-particle reduced states are maximally mixed is called AME.
- Examples: Bell states, GHZ states, quantum codewords, ...



- AME states correspond to $((N, 1, \lfloor N/2 \rfloor + 1))_D$ quantum codes.
- If D is large enough, they exist for any N.
- Qubits: They exist for $N = 2, 3, 5, 6$ but not for $N = 4$ and $N \geq 8$.
- So what happens for $N = 7$?

Note: Not all AME states are graph states, A. Burchardt & Z. Raissi, PRA 102, 022413 (2020).

The seven qubit case

First result

There is no AME state for seven qubits.

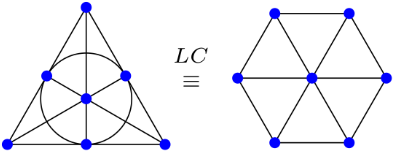
The seven qubit case

First result

There is no AME state for seven qubits.

Second result

The best approximation to a seven qubit AME state is a graph state where 32 of the 35 three-body density matrices are maximally mixed.



F. Huber et al., PRL 118, 200502 (2017).

Proof idea

(a) We use the Bloch decomposition and sort the correlations:

$$\varrho \sim \sum_{\alpha_1 \dots \alpha_n} r_{\alpha_1,\dots,\alpha_n} \sigma_{\alpha_1} \otimes \dots \otimes \sigma_{\alpha_N} \sim \left(\mathbb{1}^{\otimes n} + \sum_{j=1}^{N} P_j \right).$$

Proof idea

(a) We use the Bloch decomposition and sort the correlations:

$$\varrho \sim \sum_{\alpha_1 \ldots \alpha_n} r_{\alpha_1,\ldots,\alpha_n} \sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_N} \sim \left(\mathbb{1}^{\otimes n} + \sum_{j=1}^{N} P_j\right).$$

(b) From the Schmidt decomposition of a 7-qubit AME state $\varrho = |\phi\rangle\langle\phi|$ it follows for the five-qubit reductions

$$\varrho_{(5)}^2 = \frac{1}{4}\varrho_{(5)} \ .$$

and

$$\varrho_{(4)} \otimes \mathbb{1}^{\otimes 3} |\phi\rangle = \frac{1}{8}|\phi\rangle \quad \text{and} \quad \varrho_{(5)} \otimes \mathbb{1}^{\otimes 2} |\phi\rangle = \frac{1}{4}|\phi\rangle \, .$$

Proof idea

(a) We use the Bloch decomposition and sort the correlations:

$$\varrho \sim \sum_{\alpha_1 \ldots \alpha_n} r_{\alpha_1,\ldots,\alpha_n} \sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_N} \sim \left(\mathbb{1}^{\otimes n} + \sum_{j=1}^{N} P_j \right).$$

(b) From the Schmidt decomposition of a 7-qubit AME state $\varrho = |\phi\rangle\langle\phi|$ it follows for the five-qubit reductions

$$\varrho_{(5)}^2 = \frac{1}{4} \varrho_{(5)} .$$

and

$$\varrho_{(4)} \otimes \mathbb{1}^{\otimes 3} |\phi\rangle = \frac{1}{8} |\phi\rangle \quad \text{and} \quad \varrho_{(5)} \otimes \mathbb{1}^{\otimes 2} |\phi\rangle = \frac{1}{4} |\phi\rangle .$$

(c) Inserting this in the Bloch picture and using the commutation relation of the Paulis leads to a contradiction.

F. Huber et al., PRL 118, 200502 (2017).

General strategies

Rains' shadow inequality

Consider positive operators X and Y on N particles and $T \subset \{1, \ldots, N\}$. Then:

$$\sum_{S \subset \{1,\ldots,N\}} (-1)^{|S \cap T|} Tr_S \left[Tr_{S^c}(X) Tr_{S^c}(Y) \right] \geq 0$$

General strategies

Rains' shadow inequality

Consider positive operators X and Y on N particles and $T \subset \{1, \ldots, N\}$. Then:

$$\sum_{S \subset \{1,\ldots,N\}} (-1)^{|S \cap T|} Tr_S \left[Tr_{S^c}(X) Tr_{S^c}(Y) \right] \geq 0$$

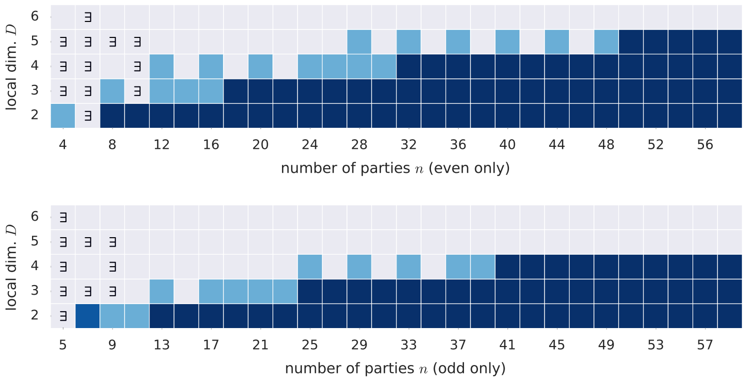
Application to the AME problem

- Assume that an AME state $|\psi\rangle$ exists and set $X = Y = |\psi\rangle\langle\psi|$.
- Since $|\psi\rangle$ is AME, many $\left[Tr_{S^c}(X)^2 \right]$ in the SI are known as proportional to the identity.
- If one finds a contradiction, the AME does not exist.

E.M. Rains, IEEE Trans. Inf. Theory 46, 54 (2000); F. Huber et al., JPA 51, 175301 (2018)

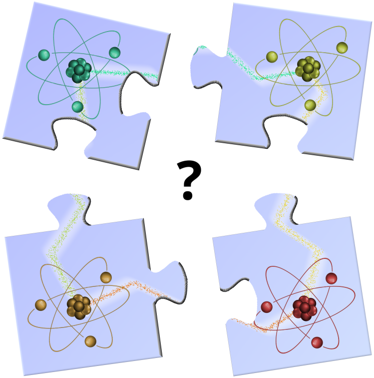
General results

Using similar ideas and the theory of weight and shadow enumerators one can exclude many more cases:

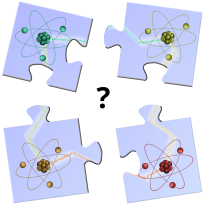


F. Huber et al., JPA 51, 175301 (2018), see also https://www.tp.nt.uni-siegen.de/+fhuber/ame.html
Recent progress: AME(4,6) exists, S.A. Rather et al., arXiv:2104.05122.

General approach to the marginal problem



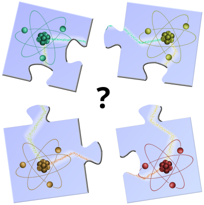
The problem



Find a pure n-particle state $|\varphi\rangle$ for some given marginals ϱ_I:

$$\text{find: } |\varphi\rangle$$
$$\text{subject to: } Tr_{I^c}(|\varphi\rangle\langle\varphi|) = \varrho_I, \;\; I \subset \{1, ..., n\}.$$

The problem



Find a pure n-particle state $|\varphi\rangle$ for some given marginals ϱ_I:

$$\text{find: } |\varphi\rangle$$
$$\text{subject to: } Tr_{I^c}(|\varphi\rangle\langle\varphi|) = \varrho_I, \; I \subset \{1, ..., n\}.$$

- If the marginals I are not overlapping: Only the eigenvalues of the ϱ_I matter, a solution is known.
  A. Klyachko, quant-ph/0409113
- The AME problem is a special case of it: $\varrho_I \sim \mathbb{1}$

Compatible states

The set of compatible states is given by

$$\mathcal{C} = \{\varrho \mid \varrho \geq 0,\ Tr_{I^c}(\varrho) = \varrho_I\ \forall I\}.$$

Question: Does $\mathcal{C}$ contain a pure state?

Compatible states

The set of compatible states is given by

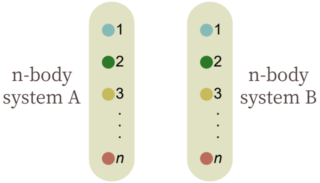
$$\mathcal{C} = \{\varrho \mid \varrho \geq 0,\ Tr_{I^c}(\varrho) = \varrho_I\ \forall I\}.$$

Question: Does $\mathcal{C}$ contain a pure state?

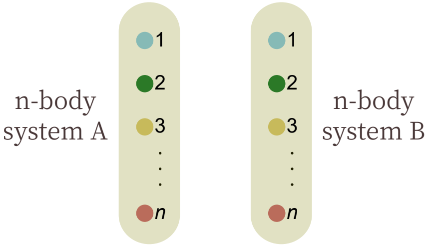
Trick

Take the convex hull of two copies of the compatible states:

$$\mathcal{C}_2 = \mathrm{conv}\{\varrho \otimes \varrho \mid \varrho \in \mathcal{C}\} = \Big\{ \sum_k p_k \varrho_k \otimes \varrho_k \mid \varrho_k \in \mathcal{C} \Big\},$$



The purity constraint

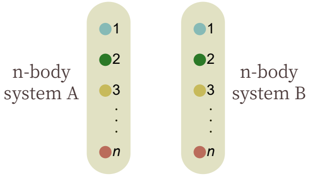


- If F_{AB} is the flip operator, then $Tr(F_{AB}\varrho_A \otimes \varrho_B) = Tr(\varrho_A\varrho_B)$.
- So, for $\Phi_{AB} \in \mathcal{C}_2$:

$$Tr(F_{AB}\Phi_{AB}) = \sum_k p_k Tr(\varrho_k^2) \leq 1.$$

- Equality holds if and only if there is a pure state in $\mathcal{C}$.

First main result



There exists are pure global state for the marginal problem if and only if the result of the following optimization equals one:

$$\max_{\Phi_{AB}} Tr(F_{AB}\Phi_{AB})$$

subject to: Φ_{AB} is separable and normalized,

$$Tr_{A_{I^c},B_{I^c}}(\Phi_{AB}) = \varrho_I \otimes \varrho_I.$$

Remains to show: If Φ_{AB} obeys the marginal condition, then all (pure!) terms in the convex combination do it also.
X.-D. Yu et al., Nature Comm. 12, 1012 (2021).

Remarks

- If $Tr(F_{AB}\Phi_{AB}) = 1$, then Φ_{AB} acts on the symmetric subspace only.

Remarks

- If $Tr(F_{AB}\Phi_{AB}) = 1$, then Φ_{AB} acts on the symmetric subspace only.
- For characterizing separability, it is convenient to go to more copies:

$$\varrho_{AB} = \sum_k p_k |a_k\rangle\langle a_k| \otimes |b_k\rangle\langle b_k| \text{ is separable}$$

$$\Rightarrow \quad \varrho_{ABB'} = \sum_k p_k |a_k\rangle\langle a_k| \otimes |b_k\rangle\langle b_k| \otimes |b_k\rangle\langle b_k| \text{ exists!}$$

Remarks

- If $Tr(F_{AB}\Phi_{AB}) = 1$, then Φ_{AB} acts on the symmetric subspace only.
- For characterizing separability, it is convenient to go to more copies:

$$\varrho_{AB} = \sum_k p_k |a_k\rangle\langle a_k| \otimes |b_k\rangle\langle b_k| \text{ is separable}$$
$$\Rightarrow \quad \varrho_{ABB'} = \sum_k p_k |a_k\rangle\langle a_k| \otimes |b_k\rangle\langle b_k| \otimes |b_k\rangle\langle b_k| \text{ exists!}$$

- The semidefinite program

$$\begin{aligned} \text{find: } & \varrho_{ABB'} \\ \text{subject to: } & Tr_{B'}(\varrho_{ABB'}) = Tr_B(\varrho_{ABB'}) = \varrho_{AB}, \\ & \varrho_{ABB'} \geq 0, \quad Tr(\varrho_{ABB'}) = 1 \end{aligned}$$

is a test for separability of ϱ_{AB}.

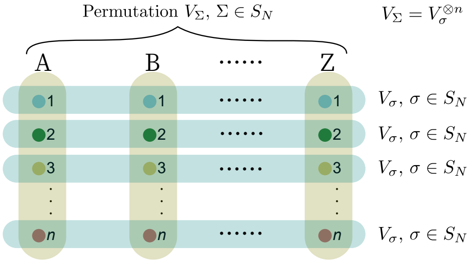
R.F. Werner, Lett. Math. Phys. 17, 359 (1989), A. C. Doherty et al., PRL 88, 187904 (2002).

The complete hierarchy

There exists are pure global state for the marginal problem if and only if for all N here exists an N-party quantum state $\Phi_{AB\cdots Z}$ such that

$$P_N^+ \Phi_{AB\cdots Z} P_N^+ = \Phi_{AB\cdots Z}$$
$$\Phi_{AB\cdots Z} \geq 0, \ \ Tr(\Phi_{AB\cdots Z}) = 1$$
$$Tr_{A_{I^c}}(\Phi_{AB\cdots Z}) = \rho_I \otimes Tr_A(\Phi_{AB\cdots Z})$$

where P_N^+ is a projector onto the symmetric space.
This is a sequence of semidefinite programs!



Symmetries & AME states

Observation

If the marginals in $Tr_{A_{I^c},B_{I^c}}(\Phi_{AB}) = \rho_I \otimes \rho_I$ obey some symmetry

$$X = gXg^{\dagger},$$

then this results in a symmetry of Φ_{AB}.
$\Rightarrow$ The set of possible Φ_{AB} becomes smaller ...

Symmetries & AME states

Observation

If the marginals in $Tr_{A_{I^c},B_{I^c}}(\Phi_{AB}) = \rho_I \otimes \rho_I$ obey some symmetry

$$X = gXg^\dagger,$$

then this results in a symmetry of Φ_{AB}.
$\Rightarrow$ The set of possible Φ_{AB} becomes smaller ...

Observation

Potential AME states have two symmetries:

- An AME state remains AME under permutation of the n particles.
- An AME state remains AME under local unitaries.

AME = Separability

Φ_{AB} is unique

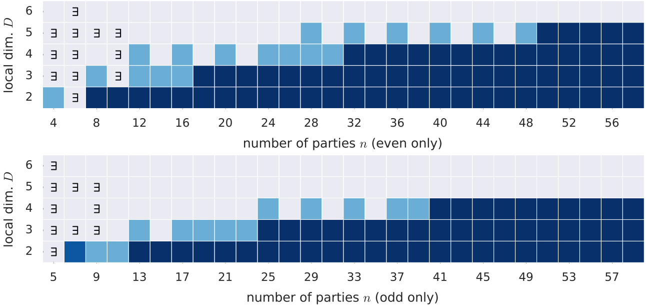
An AME(n, d) state exists if and only if an explitely given operator Φ_{AB} is a separable state w.r.t. the bipartition $(A|B)$.

AME = Separability

Φ_{AB} is unique

An AME(n, d) state exists if and only if an explitely given operator Φ_{AB} is a separable state w.r.t. the bipartition $(A|B)$.

If Φ_{AB} is not a state or NPT, the AME cannot exist.



This reproduces all known nonexistence results, apart from AME$(7, 2)$!

2021 Euros

Challenge

- Alice and Bob have four six-dimensional systems each. Let $|\phi^+\rangle = (\sum_{k=0}^{5} |kk\rangle)/\sqrt{6}$ be the maximally entangled state, define $\Pi^\perp = \mathbb{1} - |\phi^+\rangle\langle\phi^+|$.
- Then:

$$\Phi_{AB}^{T_B} = \frac{1}{1296}|\phi^+\rangle\langle\phi^+|^{\otimes 4} + \frac{1}{1587600}\left[|\phi^+\rangle\langle\phi^+|^{\otimes 1} \otimes (\Pi^\perp)^{\otimes 3} + \text{permutations}\right] + \frac{11}{18522000}\left[(\Pi^\perp)^{\otimes 4}\right].$$

- If this state is entangled, the AME$(4,6)$ does not exist.

2021 Euros

Challenge

- Alice and Bob have four six-dimensional systems each. Let $|\phi^+\rangle = (\sum_{k=0}^{5} |kk\rangle)/\sqrt{6}$ be the maximally entangled state, define $\Pi^\perp = \mathbb{1} - |\phi^+\rangle\langle\phi^+|$.
- Then:

$$\Phi_{AB}^{T_B} = \frac{1}{1296}|\phi^+\rangle\langle\phi^+|^{\otimes 4} + \frac{1}{1587600}\Big[|\phi^+\rangle\langle\phi^+|^{\otimes 1} \otimes (\Pi^\perp)^{\otimes 3} + \mathrm{permutations}\Big] + \frac{11}{18522000}\Big[(\Pi^\perp)^{\otimes 4}\Big].$$

- If this state is entangled, the AME$(4,6)$ does not exist.
- This would solve one of the "five selected open problems" in quantum information theory.

P. Horodecki, Ł. Rudnicki, K. Życzkowski, arXiv:2002.03233. Probably the money goes to: S.A. Rather et al., arXiv:2104.05122.

Training example

The seven-qubit problem

- Alice and Bob have seven qubits each. Let P_+ (P_-) projectors onto the (anti)symmetric subspace of the 2×2 system.
- Consider the state:

$$\begin{aligned}\Phi_{AB} = &\frac{113}{1119744}(P_+)^{\otimes 7} \\ &+ \frac{17}{124416}\left[(P_+)^{\otimes 5} \otimes (P_-)^{\otimes 2} + \text{permutations}\right] \\ &+ \frac{1}{13824}\left[(P_+)^{\otimes 3} \otimes (P_-)^{\otimes 4} + \text{permutations}\right] \\ &+ \frac{1}{1536}\left[(P_+)^{\otimes 1} \otimes (P_-)^{\otimes 6} + \text{permutations}\right]\end{aligned}$$

- This state is entangled, since AME$(7,2)$ does not exist.
- Can one see the entanglement directly?

Conclusion

Results

- Not all AME states exist.
- The pure state marginal problem can be solved with a hierarchy of SDPs.
- The AME problem is equivalent to a specific separability problem.

Literature

- F. Huber, O. Gühne, J. Siewert,
  Phys. Rev. Lett. 118, 200502 (2017).
- X.-D. Yu, T. Simnacher, N. Wyderka, H. C. Nguyen, O. Gühne,
  Nature Comm. 12, 1012 (2021).

Acknowledgements

DFG

House of Young Talents

DAAD

Alexander von Humboldt
Stiftung / Foundation

Chinesisch-Deutsches
Zentrum für
Wissenschaftsförderung
中德科学中心

SUOMEN KULTTUURIRAHASTO

THE ROYAL
SOCIETY

erc

Proof ingredients

(a) We use the Bloch decomposition and sort the correlations:

$$\varrho \sim \sum_{\alpha_1 \dots \alpha_n} r_{\alpha_1,\dots,\alpha_n} \sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_N} \sim \left(\mathbb{1}^{\otimes n} + \sum_{j=1}^{N} P_j \right).$$

Proof ingredients

(a) We use the Bloch decomposition and sort the correlations:

$$\varrho \sim \sum_{\alpha_1 \dots \alpha_n} r_{\alpha_1,\dots,\alpha_n} \sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_N} \sim \left(\mathbb{1}^{\otimes n} + \sum_{j=1}^{N} P_j\right).$$

(b) For anticommutators of Paulis we have the parity rule:

$$\begin{aligned} \{\sigma_x \sigma_y \sigma_z \mathbb{1}, \mathbb{1}\mathbb{1}\sigma_z\sigma_z\} &\sim \sigma_i \sigma_j \mathbb{1} \sigma_k \\ \{\mathrm{odd}, \mathrm{even}\} &\mapsto \mathrm{odd} \\ \{\mathrm{even}, \mathrm{even}\} &\mapsto \mathrm{even} \\ \{\mathrm{odd}, \mathrm{odd}\} &\mapsto \mathrm{even} \end{aligned}$$

Proof ingredients

(a) We use the Bloch decomposition and sort the correlations:

$$\varrho \sim \sum_{\alpha_1 \dots \alpha_n} r_{\alpha_1,\dots,\alpha_n} \sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_N} \sim \left(\mathbb{1}^{\otimes n} + \sum_{j=1}^{N} P_j\right).$$

(b) For anticommutators of Paulis we have the parity rule:

$$\begin{aligned} \{\sigma_x\sigma_y\sigma_z\mathbb{1}, \mathbb{1}\mathbb{1}\sigma_z\sigma_z\} &\sim \sigma_i\sigma_j\mathbb{1}\sigma_k \\ \{\text{odd}, \text{even}\} &\mapsto \text{odd} \\ \{\text{even}, \text{even}\} &\mapsto \text{even} \\ \{\text{odd}, \text{odd}\} &\mapsto \text{even} \end{aligned}$$

(c) Take a 7-qubit AME state $\varrho = |\phi\rangle\langle\phi|$. The five-qubit reduction fulfils

$$\varrho_{(5)}^2 = \frac{1}{4}\varrho_{(5)} \ .$$

and

$$\varrho_{(4)} \otimes \mathbb{1}^{\otimes 3}\,|\phi\rangle = \frac{1}{8}|\phi\rangle \quad \text{and} \quad \varrho_{(5)} \otimes \mathbb{1}^{\otimes 2}\,|\phi\rangle = \frac{1}{4}|\phi\rangle\,.$$

Proof steps

(d) Expand $\varrho_{(4)}$ and $\varrho_{(5)}$ in the Bloch basis

$$\varrho_{(4)} = \frac{1}{2^4}\left(\mathbb{1} + P_4\right), \qquad \varrho_{(5)} = \frac{1}{2^5}\left(\mathbb{1} + \sum_{j=1}^{5} P_4^{[j]} \otimes \mathbb{1}^{(j)} + P_5\right).$$

Proof steps

(d) Expand $\varrho_{(4)}$ and $\varrho_{(5)}$ in the Bloch basis

$$\varrho_{(4)} = \frac{1}{2^4}(\mathbb{1} + P_4)\,, \qquad \varrho_{(5)} = \frac{1}{2^5}(\mathbb{1} + \sum_{j=1}^{5} P_4^{[j]} \otimes \mathbb{1}^{(j)} + P_5)\,.$$

(e) Resulting eigenvalue equations:

$$P_4^{[j]} \otimes \mathbb{1}^{\otimes 3}\,|\phi\rangle = 1|\phi\rangle, \qquad P_5 \otimes \mathbb{1}^{\otimes 2}\,|\phi\rangle = 2|\phi\rangle\,.$$

Proof steps

(d) Expand $\varrho_{(4)}$ and $\varrho_{(5)}$ in the Bloch basis

$$\varrho_{(4)} = \frac{1}{2^4}(\mathbb{1} + P_4)\,, \qquad \varrho_{(5)} = \frac{1}{2^5}(\mathbb{1} + \sum_{j=1}^{5} P_4^{[j]} \otimes \mathbb{1}^{(j)} + P_5)\,.$$

(e) Resulting eigenvalue equations:

$$P_4^{[j]} \otimes \mathbb{1}^{\otimes 3}\,|\phi\rangle = 1|\phi\rangle, \qquad P_5 \otimes \mathbb{1}^{\otimes 2}\,|\phi\rangle = 2|\phi\rangle\,.$$

(f) Expanding $\varrho_{(5)}^2 = \frac{1}{4}\varrho_{(5)}$ gives two equations due to the parity rule.
One of them:

$$\{P_5, \sum_{j=1}^{5} P_4^{[j]} \otimes \mathbb{1}^{(j)}\} = 6P_5\,.$$

Proof steps

(d) Expand $\varrho_{(4)}$ and $\varrho_{(5)}$ in the Bloch basis

$$\varrho_{(4)} = \frac{1}{2^4}(\mathbb{1} + P_4)\,, \qquad \varrho_{(5)} = \frac{1}{2^5}(\mathbb{1} + \sum_{j=1}^{5} P_4^{[j]} \otimes \mathbb{1}^{(j)} + P_5)\,.$$

(e) Resulting eigenvalue equations:

$$P_4^{[j]} \otimes \mathbb{1}^{\otimes 3}\,|\phi\rangle = 1|\phi\rangle, \qquad P_5 \otimes \mathbb{1}^{\otimes 2}\,|\phi\rangle = 2|\phi\rangle\,.$$

(f) Expanding $\varrho_{(5)}^2 = \frac{1}{4}\varrho_{(5)}$ gives two equations due to the parity rule.
One of them:

$$\{P_5, \sum_{j=1}^{5} P_4^{[j]} \otimes \mathbb{1}^{(j)}\} = 6P_5\,.$$

(g) Multiplying with $|\phi\rangle$ from the right:

$$(2 \cdot 5 \cdot 1 + 5 \cdot 1 \cdot 2)\,|\phi\rangle = 6 \cdot 2\,|\phi\rangle\,.$$

F. Huber et al., PRL 118, 200502 (2017).