On the degrees of freedom in GR

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Bilbao, 27 May, 2015

Plan

Janus-faced GR:

The arena and the phenomena :

All the pre-GR physical theories provide a distinction between the **arena** in which physical phenomena take place and the **phenomena** themselves.

	arena:	phenomena:
classical mechanics	phase space: δ_{ab}	dynamical trajectories
electrodynamics	Minkowski spacetime: η_{ab}	evolution of F_{ab}
general relativity	curved spacetime: g_{ab}	evolution of g_{ab}

Such a clear distinction between the arena and the phenomenon is simply not available in general relativity

the metric plays both roles.

- GR is more than merely a field theoretic description of gravity. It is a certain body of universal rules:
  - modeling the space of events by a four-dimensional differentiable manifold
  - the use of tensor fields and tensor equations to describe physical phenomena
  - use of the (otherwise dynamical) metric in measuring of distances, areas, volumes, angles ...

The degrees of freedom in GR:

What are the degrees of freedom?

- in a theory possessing an initial value formulation: "degrees of freedom" is a synonym of "how many" distinct solutions of the equations exist
- in ordinary particle mechanics: the number of degrees of freedom is the number of quantities that must be specified as initial data divided by two

The degrees of freedom in the linearized theory:

Einstein (1916, 1918): the field equations involve **two degrees of freedom per spacetime point** when studying linearized theory

Is the full nonlinear theory characterized by two degrees of freedom?

Darmois (1927): probably the earliest **answer in the confirmatory** based on consideration of the Cauchy (or initial value) problem in GR

How to identify these two degrees of freedom?

They are supposed to be given in terms of components of the metric tensor and its derivatives or such combinations of these as, e.g. the Riemann tensor.

The degrees of freedom in GR:

What are the main issues?

- initial data: (h_{ij}, K_{ij}), metric and symmetric tensor on Σ_0

$$ {}^{(3)}R + \left(K^j{}_j\right)^2 - K_{ij}K^{ij} = 0 \ \ \& \ \ D_j K^j{}_i - D_i K^j{}_j = 0 $$

  D_i denotes the covariant derivative operator associated with h_{ij}.

- "**conformal method**" A. Lichnerowicz (1944) and J.W. York (1972): the **constraints are solved** by transforming them into a **semilinear elliptic system** by replacing the fields h_{ij} and K_{ij} by $\phi^4\,\tilde{h}_{ij}$ and $\phi^{-2}\,\tilde{K}_{ij}$...
- "... **no way singles out** precisely which functions (i.e., which of the 12 metric or extrinsic curvature components or functions of them) can be freely specified, which functions are determined by the constraints, and which functions correspond to gauge transformations. Indeed, **one of the major obstacles to developing a quantum theory of gravity** is the inability to single out the physical degrees of freedom of the theory. "

  R.M. Wald: General Relativity, Univ. Chicago Press, (1984)

- The main issue is **not to find the only legitimate quantities** representing the gravitational degrees of freedom, rather, **finding a particularly convenient embodiment** of this information (solving various problems).

The outline:

Based on some recent papers

- I. Rácz: *Is the Bianchi identity always hyperbolic?*, Class. Quantum Grav. 31 (2014) 155004
- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, Class. Quantum Grav. 32 (2015) 015006
- I. Rácz: *Dynamical determination of the gravitational degrees of freedom*, submitted to Class. Quantum Grav.
- I. Rácz and J. Winicour: *Black hole initial data without elliptic equations*, to appear in Phys. Rev. D

The main message:

1. **Euclidean and Lorentzian signature** Einsteinian spaces of $\mathbf{n+1}$-dimension $(\mathbf{n} \geq \mathbf{3})$, satisfying some mild topological assumptions, will be considered.
2. the **Bianchi identity** can be used **to explore relations** of various subsets of the basic field equations
3. new method in solving the constraints: **as opposed to the** "**conformal one**" by introducing some **geometrically distinguished variables !!! regardless** whether the primary space is Riemannian or Lorentzian
   - **momentum constraint** as a first order symmetric hyperbolic system.
   - **the Hamiltonian constraint** as a parabolic or an algebraic equation
4. the **conformal structure** appears to provide a **convenient embodiment** of the degrees of freedom

Assumptions:

- **The primary space:** (M, g_{ab})
  - M : $n+1$-dim. $(n \geq 3)$, smooth, paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian$_{(-,+,\ldots,+)}$ or Riemannian$_{(+,\ldots,+)}$ metric
- **Einsteinian space:** Einstein's equation restricting the geometry

$$\boxed{G_{ab} - \mathscr{G}_{ab} = 0}$$

with source term $\mathscr{G}_{ab}$ having a vanishing divergence, $\nabla^a \mathscr{G}_{ab} = 0$.

- or, in a more conventionally looking setup

$$\boxed{[R_{ab} - \tfrac{1}{2}\, g_{ab}\, R] + \Lambda\, g_{ab} = 8\pi\, T_{ab}}$$

with matter fields satisfying their field equations with energy-momentum tensor T_{ab} and with cosmological constant Λ

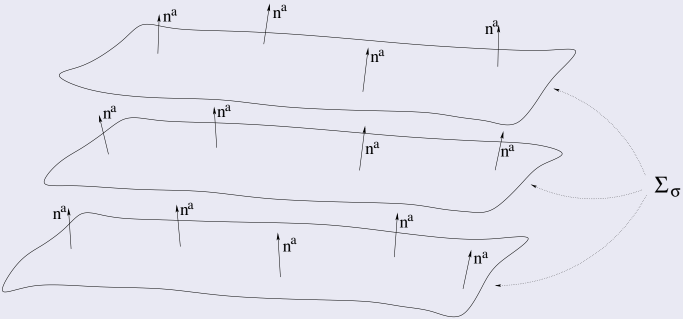
-

$$\boxed{\mathscr{G}_{ab} = 8\pi\, T_{ab} - \Lambda\, g_{ab}}$$

The primary $1+n$ splitting:

No restriction on the topology by Einstein's equations! (local PDEs)

- **Assume:** M is foliated by a one-parameter family of homologous hypersurfaces, i.e. $M \simeq \mathbb{R} \times \Sigma$, for some codimension one manifold Σ.
  - known to **hold for globally hyperbolic spacetimes** (Lorentzian case)
  - **equivalent to** the existence of a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing gradient $\nabla_a \sigma$ such that the $\sigma = const$ level surfaces $\Sigma_\sigma = \{\sigma\} \times \Sigma$ comprise the one-parameter foliation of M.



Projections:

The projection operator:

- n^a the 'unit norm' vector field that is normal to the Σ_σ level surfaces

$$\boxed{n^a n_a = \epsilon}$$

  - the sign of the norm of n^a is not fixed. ϵ takes the value -1 or $+1$ for Lorentzian or Riemannian metric g_{ab}, respectively.

- **the projection operator**

$$\boxed{h^a{}_b = \delta^a{}_b - \epsilon\, n^a n_b}$$

  to the level surfaces of $\sigma : M \to \mathbb{R}$.

- **the induced metric** on the $\sigma = const$ level surfaces

$$\boxed{h_{ab} = h^e{}_a h^f{}_b\, g_{ef}}$$

  while $\boxed{D_a}$ denotes the covariant derivative operator associated with h_{ab}.

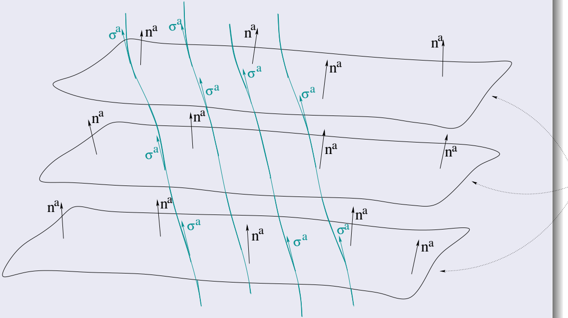
Decompositions of various fields:

Examples:

- a form field: $$L_a = \delta^e{}_a\, L_e = (h^e{}_a + \epsilon\, n^e n_a)\, L_e = \boldsymbol{\lambda}\, n_a + \mathbf{L}_a$$
    - where $\boldsymbol{\lambda} = \epsilon\, n^e\, L_e \quad \text{and} \quad \mathbf{L}_a = h^e{}_a\, L_e$
- "time evolution vector field"

$$\sigma^a : \quad \sigma^e \nabla_e \sigma = 1$$

$$\sigma^a = \sigma^a_\perp + \sigma^a_\parallel = N\, n^a + N^a$$



- where N and N^a denotes the 'laps' and 'shift' of $\sigma^a = (\partial_\sigma)^a$:

$$N = \epsilon\,(\sigma^e n_e) \quad \text{and} \quad N^a = h^a{}_e\, \sigma^e$$

Decompositions of various fields:

Any symmetric tensor field P_{ab} can be decomposed

in terms of n^a and fields living on the $\sigma = const$ level surfaces as

$$P_{ab} = \boldsymbol{\pi}\, n_a n_b + [n_a\, \mathbf{p}_b + n_b\, \mathbf{p}_a] + \mathbf{P}_{ab}$$

where $\boldsymbol{\pi} = n^e n^f\, P_{ef}\,, \quad \mathbf{p}_a = \epsilon\, h^e{}_a n^f\, P_{ef}\,, \quad \mathbf{P}_{ab} = h^e{}_a h^f{}_b\, P_{ef}$

It is also rewarding to inspect the decomposition of the contraction $\nabla^a P_{ab}$:

$$\epsilon\, (\nabla^a P_{ae})\, n^e = \mathscr{L}_n \boldsymbol{\pi} + D^e \mathbf{p}_e + [\boldsymbol{\pi}\, (K^e{}_e) - \epsilon\, \mathbf{P}_{ef} K^{ef} - 2\,\epsilon\, \dot{n}^e \mathbf{p}_e]$$
$$(\nabla^a P_{ae})\, h^e{}_b = \mathscr{L}_n \mathbf{p}_b + D^e \mathbf{P}_{eb} + [(K^e{}_e)\, \mathbf{p}_b + \dot{n}_b\, \boldsymbol{\pi} - \epsilon\, \dot{n}^e \mathbf{P}_{eb}]$$

$$\dot{n}_a := n^e \nabla_e n_a = -\epsilon\, D_a \ln N$$

back

Decompositions of various fields:

Examples:

- the metric

$$g_{ab} = \epsilon\, n_a n_b + h_{ab}$$

- the "source term"

$$\mathscr{G}_{ab} = n_a n_b\, \mathfrak{e} + [n_a\, \mathfrak{p}_b + n_b\, \mathfrak{p}_a] + \mathfrak{S}_{ab}$$

where $\mathfrak{e} = n^e n^f\, \mathscr{G}_{ef}\,, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f\, \mathscr{G}_{ef}\,, \quad \mathfrak{S}_{ab} = h^e{}_a h^f{}_b\, \mathscr{G}_{ef}$

- the r.h.s. of our basic field equation $E_{ab} = G_{ab} - \mathscr{G}_{ab}$

$$E_{ab} = n_a n_b\, E^{(\mathcal{H})} + [n_a\, E^{(\mathcal{M})}_b + n_b\, E^{(\mathcal{M})}_a] + (E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})$$

$$E^{(\mathcal{H})} = n^e n^f\, E_{ef}\,, \quad E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f\, E_{ef}\,, \quad E^{(\mathcal{EVOL})}_{ab} = h^e{}_a h^f{}_b\, E_{ef} - h_{ab}\, E^{(\mathcal{H})}$$

Relations between various parts of the basic equations:

The decomposition of the covariant divergence $\nabla^a E_{ab} = 0$ of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$:

$$\mathscr{L}_n E^{(\mathcal{H})} + D^e E^{(\mathcal{M})}_e + [\, E^{(\mathcal{H})} (K^e{}_e) - 2\,\epsilon\,(\dot n^e E^{(\mathcal{M})}_e) - \epsilon\, K^{ae}\,(E^{(\mathcal{EVOL})}_{ae} + h_{ae}\, E^{(\mathcal{H})})\,] = 0$$

back

$$\mathscr{L}_n E^{(\mathcal{M})}_b + D^a(E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})}) + [\,(K^e{}_e)\, E^{(\mathcal{M})}_b + E^{(\mathcal{H})}\,\dot n_b - \epsilon\,(E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})\,\dot n^a\,] = 0$$

a first order symmetric hyperbolic linear homogeneous system for $(E^{(\mathcal{H})}, E^{(\mathcal{M})}_i)^T$ fosh

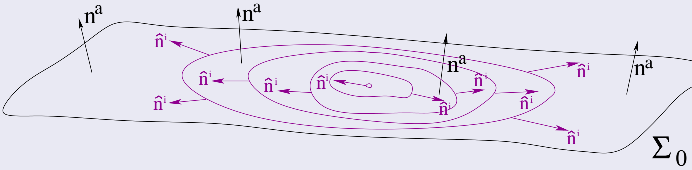
Theorem

Let (M, g_{ab}) be as specified above and assume that the metric h_{ab} induced on the $\sigma = const$ level surfaces is Riemannian. Then, **regardless whether g_{ab} is of Lorentzian or Euclidean signature**, *any solution to the reduced equations $E^{(\mathcal{EVOL})}_{ab} = 0$ is also a solution to the full set of field equations $G_{ab} - \mathscr{G}_{ab} = 0$ provided that the constraint expressions $E^{(\mathcal{H})}$ and $E^{(\mathcal{M})}_a$ vanish on one of the $\sigma = const$ level surfaces.* back

The secondary $1 + [n-1]$ splitting:

Assume now that on one of the $\sigma = const$ level surfaces—say on Σ_0—there exists a smooth function $\rho : \Sigma_0 \to \mathbb{R}$, with nowhere vanishing gradient such that:

- the $\rho = const$ level surfaces $\mathscr{S}_\rho$ are homologous to each other and such that they are orientable compact without boundary in M.



- The metric h_{ij} on Σ_0 can be decomposed as

$$h_{ij} = \hat{\gamma}_{ij} + \hat{n}_i \hat{n}_j$$

in terms of the positive definite metric $\hat{\gamma}_{ij}$, induced on the $\mathscr{S}_\rho$ hypersurfaces,

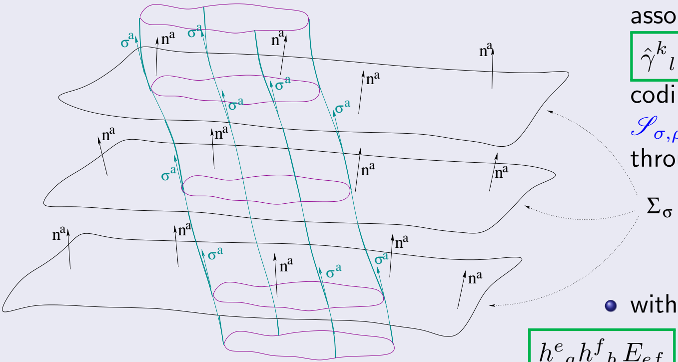
- and the unit norm field

$$\hat{n}^i = \hat{N}^{-1} [\, (\partial_\rho)^i - \hat{N}^i \,]$$

normal to the $\mathscr{S}_\rho$ hypersurfaces on Σ_0, where $\hat{N}$ and $\hat{N}^i$ denotes the 'laps' and 'shift' of an 'evolution' vector field $\rho^i = (\partial_\rho)^i$ on Σ_0.

Secondary projections:

The Lie transport of this foliation of Σ_0 along the integral curves of the vector field σ^a yields then a two-parameter foliation $\mathscr{S}_{\sigma,\rho}$:



- the fields $\hat{n}^i$, $\hat{\gamma}_{ij}$ and the associated projection op. $\boxed{\hat{\gamma}^k{}_l = h^k{}_l - \hat{n}^k \hat{n}_l}$ to the codimension-two surfaces $\mathscr{S}_{\sigma,\rho}$ get to be well-defined throughout M.

- with some algebra

$$\boxed{h^e{}_a h^f{}_b\, E_{ef}} = E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})}$$

- can be put into the form

$$h^e{}_i h^f{}_j\, E_{ef} = \boxed{{}^{(n)}E_{ij} = {}^{(n)}G_{ij} - {}^{(n)}\mathscr{G}_{ij}}$$

The integrability condition for ${}^{(n)}G_{ij} - {}^{(n)}\mathscr{G}_{ij} = 0$

$$ {}^{(n)}E_{ij} = \hat{E}^{(\mathcal{H})}\hat{n}_i\hat{n}_j + [\hat{n}_i\hat{E}_j^{(\mathcal{M})} + \hat{n}_j\hat{E}_i^{(\mathcal{M})}] + (\hat{E}_{ij}^{(\mathcal{EVOL})} + \hat{\gamma}_{ij}\hat{E}^{(\mathcal{H})}) $$

$$ \hat{E}^{(\mathcal{H})} = \hat{n}^e\hat{n}^f\,{}^{(n)}E_{ef}, \quad \hat{E}_i^{(\mathcal{M})} = \hat{\gamma}^e{}_j\hat{n}^f\,{}^{(n)}E_{ef}, \quad \hat{E}_{ij}^{(\mathcal{EVOL})} = \hat{\gamma}^e{}_i\hat{\gamma}^f{}_j\,{}^{(n)}E_{ef} - \hat{\gamma}_{ij}\hat{E}^{(\mathcal{H})} $$

Lemma

The integrability condition $D^i[{}^{(n)}\mathscr{G}_{ij}] = 0$ holds on Σ_σ if the momentum constraint expression $E_b^{(\mathcal{M})}$, along with its Lie derivative $\mathscr{L}_n E_b^{(\mathcal{M})}$, vanishes there.

Corollary

Assume that $E_b^{(\mathcal{M})} = 0$ on all the Σ_σ level surfaces, and that both $\hat{E}^{(\mathcal{H})}$ and $\hat{E}_a^{(\mathcal{M})}$ vanish along a world-tube $\mathscr{W}_\mathscr{S}$ in M. Then any solution to the secondary reduced equations $\hat{E}_{ij}^{(\mathcal{EVOL})} = 0$ is also a solution to the secondary equations ${}^{(n)}G_{ij} - {}^{(n)}\mathscr{G}_{ij} = 0$.

◂ Theorem

Corollary

Assume, in addition, that $E^{(\mathcal{H})} = 0$ on Σ_0. Then any solution to the reduced equations $\hat{E}_{ij}^{(\mathcal{EVOL})} = 0$ is also a solution to the original basic field equations $G_{ij} - \mathscr{G}_{ij} = 0$.

$[\ E_b^{(\mathcal{M})} = 0$ on $\Sigma_0 \iff \{\hat{E}^{(\mathcal{H})} = 0, \hat{E}_i^{(\mathcal{M})} = 0\}$ on $\mathscr{W}_\mathscr{S}\ ]$

The explicit forms:

Expressions in the $1+n$ decomposition:

$$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\,{}^{(n)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\}$$

$$E^{(\mathcal{M})}_a = h^e{}_a n^f E_{ef} = D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a$$

$$E^{(\mathcal{EVOL})}_{ab} = {}^{(n)}\!R_{ab} + \epsilon\,\{-\mathscr{L}_n K_{ab} - (K^e{}_e)K_{ab} + 2\,K_{ae}K^e{}_b - \epsilon\,N^{-1}D_aD_bN\} + \tfrac{1+\epsilon}{(n-1)}\,h_{ab}\,E^{(\mathcal{H})} - \Big(\mathfrak{S}_{ab} - \tfrac{1}{n-1}\,h_{ab}\,[\mathfrak{S}_{ef}\,h^{ef} + \epsilon\,\mathfrak{e}]\Big)$$

where

$$\mathfrak{e} = n^e n^f \mathscr{G}_{ef}, \quad \mathfrak{p}_a = \epsilon\,h^e{}_a n^f \mathscr{G}_{ef}, \quad \mathfrak{S}_{ab} = h^e{}_a h^f{}_b \mathscr{G}_{ef}$$

and the extrinsic curvature K_{ab} which is defined as

$$K_{ab} = h^e{}_a \nabla_e n_b = \tfrac{1}{2}\,\mathscr{L}_n h_{ab}$$

where $\mathscr{L}_n$ stands for the Lie derivative with respect to n^a

The explicit forms:

Expressions in the $1+[n-1]$ decomposition:

$$
\begin{aligned}
\hat E^{(\mathcal{H})} &= \tfrac{1}{2}\,\{-\hat R + (\hat K^l{}_l)^2 - \hat K_{kl}\hat K^{kl} - 2\,\hat{\mathfrak{e}}\}\,, \\
\hat E^{(\mathcal{M})}_i &= \hat D^l \hat K_{li} - \hat D_i \hat K^l{}_l - \hat{\mathfrak{p}}_i\,, \\
\hat E^{(\mathcal{EVOL})}_{ij} &= \hat R_{ij} - \mathscr{L}_{\hat n}\hat K_{ij} - (\hat K^l{}_l)\hat K_{ij} + 2\,\hat K_{il}\hat K^l{}_j - \hat N^{-1}\,\hat D_i \hat D_j \hat N \\
&\quad + \hat\gamma_{ij}\{\mathscr{L}_{\hat n}\hat K^l{}_l + \hat K_{kl}\hat K^{kl} + \hat N^{-1}\hat D^l \hat D_l \hat N\} - [\hat{\mathfrak{S}}_{ij} - \hat{\mathfrak{e}}\,\hat\gamma_{ij}]
\end{aligned}
$$

where $\hat D_i, \hat R_{ij}$ and $\hat R$ denote the covariant derivative operator, the Ricci tensor and the scalar curvature of $\hat\gamma_{ij}$, respectively. The 'hatted' source terms $\hat{\mathfrak{e}}$, $\hat{\mathfrak{p}}_i$ and $\hat{\mathfrak{S}}_{ij}$ and the extrinsic curvature $\hat K_{ij}$ are defined as

$$
\hat{\mathfrak{e}} = \hat n^k \hat n^l\, {}^{(n)}\mathscr{G}_{kl}\,, \quad \hat{\mathfrak{p}}_i = \hat\gamma^k{}_i\, \hat n^l\, {}^{(n)}\mathscr{G}_{kl} \quad \text{and} \quad \hat{\mathfrak{S}}_{ij} = \hat\gamma^k{}_i \hat\gamma^l{}_j\, {}^{(n)}\mathscr{G}_{kl}
$$

and

$$
\hat K_{ij} = \hat\gamma^l{}_i\, D_l\, \hat n_j = \tfrac{1}{2}\,\mathscr{L}_{\hat n}\hat\gamma_{ij}
$$

The $1+[n-1]$ decomposition of the extrinsic curvature:

The Σ_σ hypersurfaces in both cases are spacelike:

$$K_{ij} = \boldsymbol{\kappa}\, \hat n_i \hat n_j + [\hat n_i\, \mathbf{k}_j + \hat n_j\, \mathbf{k}_i] + \mathbf{K}_{ij}$$

$$\begin{aligned}
\boldsymbol{\kappa} &= \hat n^k \hat n^l\, K_{kl} = \hat n_k (\mathscr{L}_n \hat n^k) \\
\mathbf{k}_i &= \hat\gamma^k{}_i \hat n^l\, K_{kl} = \tfrac{1}{2}\hat\gamma^k{}_i\, (\mathscr{L}_n \hat n_k) - \tfrac{1}{2}\hat\gamma_{ki}\, (\mathscr{L}_n \hat n^k) \\
\mathbf{K}_{ij} &= \hat\gamma^k{}_i \hat\gamma^l{}_j\, K_{kl} = \tfrac{1}{2}\, \hat\gamma^k{}_i \hat\gamma^l{}_j\, (\mathscr{L}_n \hat\gamma_{kl})
\end{aligned}$$

$$\mathbf{K}^l{}_l = \hat\gamma^{kl}\, \mathbf{K}_{kl} = \tfrac{1}{2}\, \hat\gamma^{ij} (\mathscr{L}_n \hat\gamma_{ij})$$

projection taking the trace free parts on the $\mathscr{S}_{\sigma,\rho}$ surfaces:

$$\Pi^{kl}{}_{ij} = \hat\gamma^k{}_i \hat\gamma^l{}_j - \tfrac{1}{n-1}\, \hat\gamma_{ij} \hat\gamma^{kl}$$

$$\mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{n-1}\, \hat\gamma_{ij} (\hat\gamma^{ef}\, \mathbf{K}_{ef})$$

The $1+n$ constraints

The momentum constraint:

$$E^{(\mathcal{M})}_a = h^e{}_a n^f E_{ef} = D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a = 0$$

div

$$(\hat{K}^l{}_l)\,\mathbf{k}_i + \hat{D}^l \mathring{\mathbf{K}}_{li} + \boldsymbol{\kappa}\,\dot{\hat{n}}_i + \mathscr{L}_{\hat{n}}\mathbf{k}_i - \dot{\hat{n}}^l\,\mathbf{K}_{li} - \hat{D}_i\boldsymbol{\kappa} - \tfrac{n-2}{n-1}\,\hat{D}_i(\mathbf{K}^l{}_l) - \epsilon\,\mathfrak{p}_l\,\hat{\gamma}^l{}_i = 0$$

$$\boldsymbol{\kappa}\,(\hat{K}^l{}_l) + \hat{D}^l\mathbf{k}_l - \mathbf{K}_{kl}\hat{K}^{kl} - 2\,\dot{\hat{n}}^l\,\mathbf{k}_l - \mathscr{L}_{\hat{n}}(\mathbf{K}^l{}_l) - \epsilon\,\mathfrak{p}_l\,\hat{n}^l = 0$$

where $\dot{\hat{n}}_k = \hat{n}^l D_l \hat{n}_k = -\hat{D}_k(\ln \hat{N})$

With some algebra in coordinates $(\rho, x^3, \dots, x^{n+1})$ adopted to the foliation $\mathscr{S}_{\sigma,\rho}$:

$$\left\{\begin{pmatrix} \frac{n-1}{(n-2)\,\hat{N}}\,\hat{\gamma}^{AB} & 0 \\ 0 & 1 \end{pmatrix}\partial_\rho + \begin{pmatrix} -\frac{(n-1)\,\hat{N}^K}{(n-2)\,\hat{N}}\,\hat{\gamma}^{AB} & -\hat{\gamma}^{AK} \\ -\hat{\gamma}^{BK} & -\hat{N}^K \end{pmatrix}\partial_K\right\}\begin{pmatrix} \mathbf{k}_B \\ \mathbf{K}^E{}_E \end{pmatrix} + \begin{pmatrix} \mathscr{B}^{A}_{(\mathbf{k})} \\ \mathscr{B}_{(\mathbf{K})} \end{pmatrix} = 0$$

Is a first order **symmetric hyperbolic** system for the vector valued variable

$$(\mathbf{k}_B, \mathbf{K}^E{}_E)^T$$

where **the 'radial coordinate' ρ plays the role of 'time'**. ... with characteristic cone (apart from the surfaces $\mathscr{S}_\rho$ with $\hat{n}^i\xi_i = 0$) $[\hat{\gamma}^{ij} - (n-1)\,\hat{n}^i\hat{n}^j]\,\xi_i\xi_j = 0$

The $1+n$ constraints

The Hamiltonian constraint:

$$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\,{}^{(n)}R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0$$

using $$ {}^{(n)}R = \hat R - \Big\{2\,\mathscr{L}_{\hat n}(\hat K^l{}_l) + (\hat K^l{}_l)^2 + \hat K_{kl}\hat K^{kl} + 2\,\hat N^{-1}\hat D^l\hat D_l\hat N\Big\}$$

$$-\epsilon\,\hat R + \epsilon\Big\{2\,\mathscr{L}_{\hat n}(\hat K^l{}_l) + (\hat K^l{}_l)^2 + \hat K_{kl}\,\hat K^{kl} + 2\,\hat N^{-1}\hat D^l\hat D_l\hat N\Big\} + 2\,\boldsymbol{\kappa}\,\mathbf{K}^l{}_l + (\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathbf{K}_{kl}\,\mathbf{K}^{kl} - 2\,\mathfrak{e} = 0$$

- algebraic equation for $\boldsymbol{\kappa}$ provided that $\mathbf{K}^l{}_l$ does not vanish
- eliminating $\boldsymbol{\kappa}$ $\Rightarrow$ the momentum constraint becomes **a strongly hyperbolic system** for $(\mathbf{k}_i, \mathbf{K}^l{}_l)^T$ **provided that** $\boldsymbol{\kappa}$ **and** $\mathbf{K}^l{}_l$ **are of opposite sign**
- by choosing the free data $(\hat N, \hat N^i, \hat\gamma_{ij}, \mathring{\mathbf{K}}_{ij})$ on Σ_0 this **can be guaranteed locally**
- considering data in **Kerr-Schild form**: $g_{ab} = \eta_{ab} + 2H\ell_a\ell_b$, ($H$ smooth**!** on $\mathbb{R}^4$, ℓ_a is null with respect to both g_{ab} and an implicit background Minkowski metric η_{ab}) for near Schwarzschild $\frac{\mathbf{k}_A}{\boldsymbol{\kappa}} \approx 0$ approximations: $-\frac{\mathbf{K}^l{}_l}{\boldsymbol{\kappa}} \approx \frac{2(1+2H)}{1+H}$ everywhere **!**

Conformal structure by splitting of the induced metric $\hat{\gamma}_{ij}$:

There exist a smooth function $\Omega : \Sigma_0 \to \mathbb{R}$—which does not vanish except at an origin where the foliation $\mathscr{S}_\rho$ smoothly reduces to a point on the Σ_0 level surfaces—such that the induced metric $\hat{\gamma}_{ij}$ can be decomposed as

$$\hat{\gamma}_{ij} = \Omega^2 \, \gamma_{ij}$$

where γ_{ij} is such that

$$\gamma^{ij}(\mathscr{L}_\rho \gamma_{ij}) = 0$$

throughout Σ_0 surfaces.

What does the second relation mean?

- In virtue of

$$\gamma^{ij}(\mathscr{L}_\rho \gamma_{ij}) = \mathscr{L}_\rho \, \ln[\det(\gamma_{ij})]$$

  the determinant is independent of ρ but may depend on the 'angular' coordinates.
- Does the desired smooth function $\Omega : \Sigma_0 \to \mathbb{R}$ and the metric γ_{ij} exist?

The conformal structure: $\gamma_{ij} = \Omega^{-2}\,\hat{\gamma}_{ij}$

The construction of $\Omega : \Sigma_0 \to \mathbb{R}$:

- for any smooth distribution of the induced metric $\hat{\gamma}_{ij}$ one may integrate

$$\hat{\gamma}^{ij}(\mathscr{L}_\rho \hat{\gamma}_{ij}) = \cancel{\gamma^{ij}(\mathscr{L}_\rho \gamma_{ij})} + (n-1)\,\mathscr{L}_\rho(\ln \Omega^2)$$

  along the integral curves of ρ^a on Σ_0, starting with a smooth non-vanishing function $\Omega_0 = \Omega_0(x^3, \dots, x^{n+1})$ at $\mathscr{S}_0$.
- $\Omega^2 = \Omega^2(\rho, x^3, \dots, x^{n+1})$ can be given as

$$\Omega^2 = \Omega_0^2 \cdot \exp\left[\tfrac{1}{n-1}\int_0^\rho \left(\hat{\gamma}^{ij}(\mathscr{L}_\rho \hat{\gamma}_{ij})\right)\, d\tilde{\rho}\right]$$

The conformal structure satisfying $\gamma^{ij}(\mathscr{L}_\rho \gamma_{ij}) = 0$ can be given then as:

$$\gamma_{ij} = \Omega^{-2}\,\hat{\gamma}_{ij}$$

The other faces of the Hamiltonian constraint:

$$-\epsilon\,\hat R + \epsilon\left\{2\,\mathscr{L}_{\hat n}(\hat K^l{}_l)\;+(\hat K^l{}_l)^2 + \hat K_{kl}\,\hat K^{kl} + 2\,\hat N^{-1}\,\hat D^l\,\hat D_l \hat N\right\} + 2\,\boldsymbol{\kappa}\,\mathbf{K}^l{}_l + (\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathbf{K}_{kl}\,\mathbf{K}^{kl} - 2\,\mathfrak{e} = 0$$

- $\not\epsilon = \pm 1$ elliptic equation for Ω: using $\hat K^l{}_l = \frac{n-1}{2}\,\mathscr{L}_{\hat n}\ln\Omega^2 - \hat N^{-1}\,\mathbb{D}_k\hat N^k$ and

$$\hat\gamma_{ij} = \Omega^2\,\gamma_{ij} \Longrightarrow \hat R = \Omega^{-2}\left[{}^{(\gamma)}R - (n-2)\left\{\mathbb{D}^l\mathbb{D}_l\ln\Omega^2 + \frac{(n-3)}{4}(\mathbb{D}^l\ln\Omega^2)(\mathbb{D}_l\ln\Omega^2)\right\}\right]$$

- parabolic equation for $\hat N$:

$$\hat K^l{}_l = \hat N^{-1}\,[\tfrac{n-1}{2}\,\mathscr{L}_\rho\ln\Omega^2 - \hat D_k\hat N^k]\,,\;\; \mathscr{L}_{\hat n}(\hat K^l{}_l) = [\ldots]\cdot\mathscr{L}_{\hat n}\hat N + \ldots\;\&\;\hat N^{-1}\,\hat D^l\,\hat D_l\hat N$$

R. Bartnik (1993), R. Weinstein & B. Smith (2004)

Sorting the components of (h_{ij}, K_{ij}):

- The twelve independent components of the pair (h_{ij}, K_{ij}) may be represented by

$$(\hat{N}, \hat{N}^i, \Omega, \gamma_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$$

  - or by applying

$$\boldsymbol{\kappa} = \mathscr{L}_n \ln \hat{N} \quad \text{and} \quad \mathbf{k}_i = (2\hat{N})^{-1}\, \hat{\gamma}_{il}\, (\mathscr{L}_n \hat{N}^l)$$

$$\mathbf{K}^l{}_l = \tfrac{n-1}{2}\, \mathcal{L}_n \ln \Omega^2 \quad \text{and} \quad \mathring{\mathbf{K}}_{ij} = \tfrac{1}{2}\, \Omega^2\, \gamma^k{}_i \gamma^l{}_j\, (\mathscr{L}_n \gamma_{kl})$$

$$(\hat{N}, \hat{N}^i, \Omega, \gamma_{ij}; \mathscr{L}_n \hat{N}, \mathscr{L}_n \hat{N}^l, \mathcal{L}_n \Omega, \mathscr{L}_n \gamma_{ij})$$

  - The momentum constraint (satisfying a hyperbolic system) can always be solved as an initial value problem with initial data specified at some $\mathscr{S}_\rho \subset \Sigma_\sigma$ for the variables $\mathscr{L}_n \hat{N}^l, \mathcal{L}_n \Omega$.
  - The Hamiltonian constraint:
    - $\not{\epsilon} = \pm 1$ elliptic equation for Ω: ill-posed together with the hyp.mom.constr.
    - parabolic equation for $\hat{N}$:
      freely specifiable: $(\hat{N}^i, \Omega, \boldsymbol{\gamma_{ij}}; \mathscr{L}_n \hat{N}, \boldsymbol{\mathscr{L}_n \gamma_{ij}})$
    - algebraic equation for $\boldsymbol{\kappa}$:
      freely specifiable: $(\hat{N}, \hat{N}^i, \Omega, \boldsymbol{\gamma_{ij}}; \boldsymbol{\mathscr{L}_n \gamma_{ij}})$

Summary:

1. **Euclidean and Lorentzian signature** Einsteinian spaces of $\mathbf{n+1}$-dimension $(\mathbf{n \geq 3})$ were considered. **The topology of** M was restricted by assuming:
   - smoothly foliated by a one-parameter family of homologous hypersurfaces
   - one of these level surfaces is smoothly foliated by a one-parameter family of codimension-two-surfaces (orientable compact without boundary in M)
2. the **Bianchi identity** and **a pair of nested decompositions** can be used **to explore relations** of various projections of the field equations
3. solving the $\mathbf{1+n}$ constraints: by introducing some **geometrically distinguished variables !!! regardless** whether the primary space is Riemannian or Lorentzian
   - **momentum constraint** as a **first order symmetric hyperbolic system**.
   - **the Hamiltonian constraint** as a **parabolic** or an **algebraic** equation
4. the **conformal structure** γ_{ij}, defined on the foliating codimension-two surfaces $\mathscr{S}_\rho$, appears to provide a **convenient embodiment** of the $\frac{(n-1)\,n}{2} - 1$ degrees of freedom to various metric theories of gravity

Thanks for your attention

First order symmetric hyperbolic linear homogeneous system for $(E^{(\mathcal{H})}, E_i^{(\mathcal{M})})^T$:

$$\mathscr{L}_n E^{(\mathcal{H})} + D^e E_e^{(\mathcal{M})} + [\, E^{(\mathcal{H})}\,(K^e{}_e) - 2\,\epsilon\,(\dot n^e\, E_e^{(\mathcal{M})}) - \epsilon\, K^{ae}\,(E_{ae}^{(\mathcal{EVOL})} + h_{ae}\, E^{(\mathcal{H})})\,] = 0$$

$$\mathscr{L}_n E_b^{(\mathcal{M})} + D^a (E_{ab}^{(\mathcal{EVOL})} + h_{ab}\, E^{(\mathcal{H})}) + [\,(K^e{}_e)\, E_b^{(\mathcal{M})} + E^{(\mathcal{H})}\,\dot n_b - \epsilon\,(E_{ab}^{(\mathcal{EVOL})} + h_{ab}\, E^{(\mathcal{H})})\,\dot n^a\,] = 0$$

- When writing them out explicitly in some local coordinates $(\sigma, x^1, \ldots, x^n)$ adopted to the vector field $\sigma^a = N\, n^a + N^a: \quad \sigma^e \nabla_e \sigma = 1$ and the foliation $\{\Sigma_\sigma\}$, read as

$$\left\{ \begin{pmatrix} \frac{1}{N} & 0 \\ 0 & \frac{1}{N}\, h^{ij} \end{pmatrix} \partial_\sigma + \begin{pmatrix} -\frac{1}{N}\, N^k & h^{ik} \\ h^{jk} & -\frac{1}{N}\, N^k\, h^{ij} \end{pmatrix} \partial_k \right\} \begin{pmatrix} E^{(\mathcal{H})} \\ E_i^{(\mathcal{M})} \end{pmatrix} = \begin{pmatrix} \mathscr{E} \\ \mathscr{E}^j \end{pmatrix}$$

where the source terms $\mathscr{E}$ and $\mathscr{E}^j$ are linear and homogeneous in $E^{(\mathcal{H})}$ and $E_i^{(\mathcal{M})}$. back

- It is also informative to inspect the characteristic cone associated with the above equation which—apart from the hypersurfaces Σ_σ with $n^i \xi_i = 0$—can be given as

$$(h^{ij} - n^i n^j)\, \xi_i \xi_j = 0$$

Relations between various parts of the basic equations:

Corollary

If the constraint expressions $E^{(\mathcal{H})}$ *and* $E_a^{(\mathcal{M})}$ *vanish on all the* $\sigma = const$ *level surfaces then the relations*

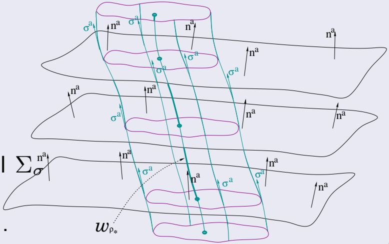
$$\begin{aligned} K^{ab}\, E^{(\mathcal{EVOL})}_{ab} &= 0 \\ D^a E^{(\mathcal{EVOL})}_{ab} - \epsilon\, \dot{n}^a\, E^{(\mathcal{EVOL})}_{ab} &= 0 \end{aligned}$$

hold for the evolutionary expression $E^{(\mathcal{EVOL})}_{ab}$.

Having an origin

A world-line $\mathscr{W}_{\rho_*}$ represents an origin in M:

- If the foliating codimension-two-surfaces smoothly reduce to a point on the Σ_σ level surfaces at the location $\rho = \rho_*$. back
  - Note that then Ω vanishes at $\rho = \rho_*$. $\Longrightarrow$
  - The existence of an origin on the individual Σ_σ level surfaces is signified by the limiting behavior $\hat{\gamma}^{ij}(\mathscr{L}_\rho \hat{\gamma}_{ij}) \to \pm\infty$ while $\rho \to \rho_*^{\pm}$.



To have a regular origin in M:

- One needs to impose further conditions excluding the occurrence of various defects such as the existence of a conical singularity.
  - An origin $\mathscr{W}_{\rho_*}$ will be referred as being **regular** if there exist smooth functions $\hat{N}_{(2)}, \Omega_{(3)}$ and $\hat{N}^A_{(1)}$ such that, in a neighborhood of the location $\rho = \rho_*$ on the Σ_σ level surfaces, the basic variables $\hat{N}, \Omega$ and $\hat{N}^A$ can be given as

$$\hat{N} = 1 + (\rho - \rho_*)^2\, \hat{N}_{(2)},\ \ \Omega = (\rho - \rho_*) + (\rho - \rho_*)^3\, \Omega_{(3)},\ \ \hat{N}^A = (\rho - \rho_*)\, \hat{N}^A_{(1)}$$