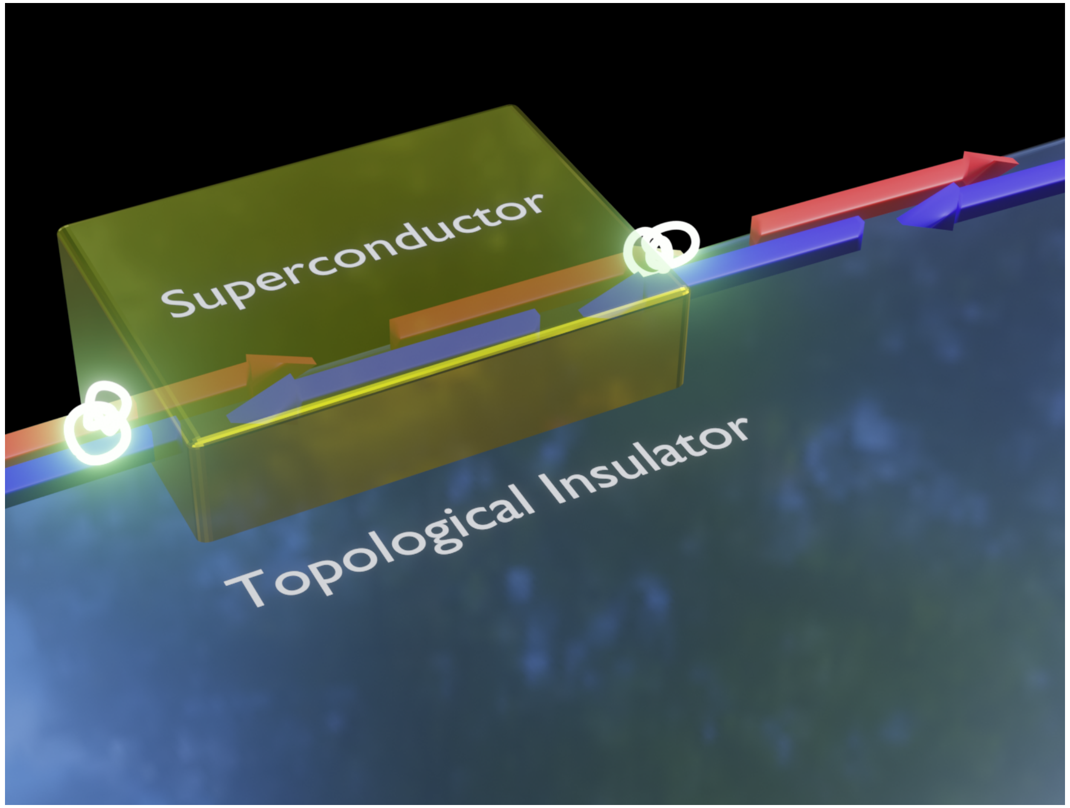
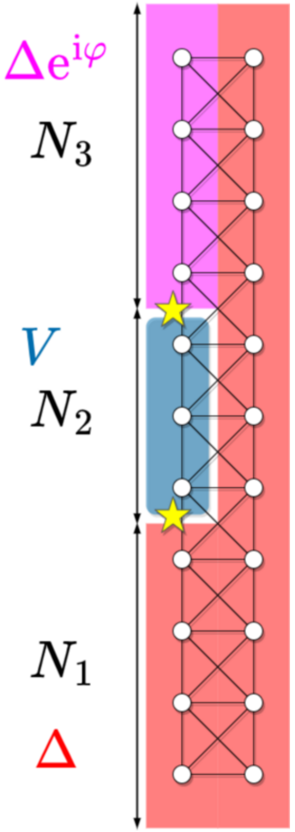
A simple electronic ladder model harboring $\mathbb{Z}_4$ parafermions





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Botond Osváth
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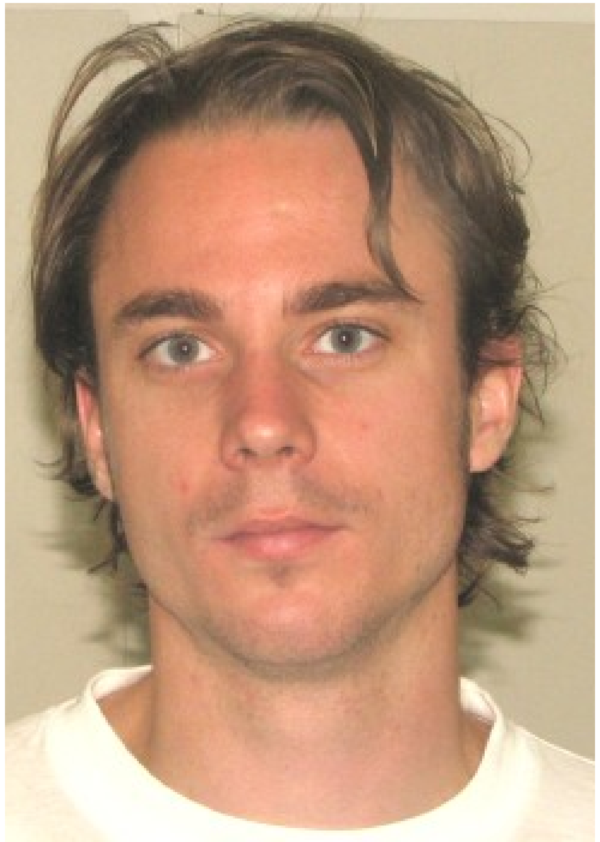
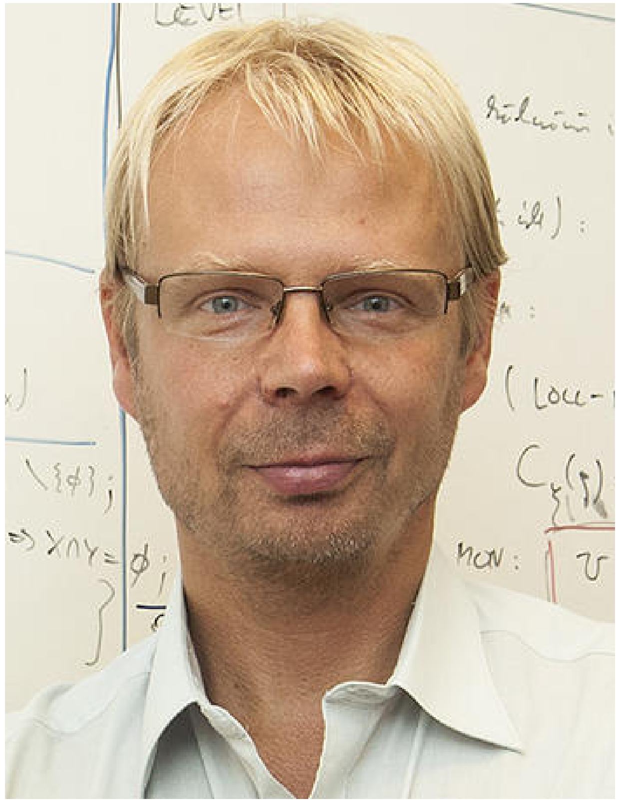


Gergely Barcza
Wigner FK

Balázs Dóra
BME



Örs Legeza
Wigner FK

Outline

- **What** are Parafermions ? **Why** should you care?
- Where **did people** look for parafermions?
- Where **we** look for parafermions ?
- Where **one could** look for parafermions ?

Clock models and parafermions

$$H = -J\sum_{p=1}^{L-1}\hat{\sigma}_p^\dagger\hat{\sigma}_{p+1} - f\sum_{p=1}^{L}\hat{\tau}_p + \mathrm{h.c.}$$

N=3 Clock model

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Omega & 0 \\ 0 & 0 & \Omega^2 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Omega = \mathrm{e}^{\mathrm{i}2\pi/N} = \omega^2$$

Clock models and parafermions

$$H = -J \sum_{p=1}^{L-1} \hat{\sigma}_p^\dagger \hat{\sigma}_{p+1} - f \sum_{p=1}^{L} \hat{\tau}_p + \mathrm{h.c.}$$

$$\Updownarrow$$

$$H = -J \sum_{p=1}^{L-1} \omega \hat{\alpha}_{2p}^\dagger \hat{\alpha}_{2p+1} - f \sum_{p=1}^{L} \omega \hat{\alpha}_{2p-1}^\dagger \hat{\alpha}_{2p}$$

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Jordan-Wigner

$$\hat{\alpha}_{2p-1} = \hat{\sigma}_p \prod_{q<p} \hat{\tau}_q$$

$$\hat{\alpha}_{2p} = -\omega \hat{\tau}_p \hat{\sigma}_p \prod_{q<p} \hat{\tau}_q$$

Parafermion

$$\hat{\alpha}_p^N = \hat{1}, \hat{\alpha}_p^\dagger = \hat{\alpha}_p^{N-1}$$

$$\hat{\alpha}_p \hat{\alpha}_q = \Omega^{\mathrm{sign}(q-p)} \hat{\alpha}_q \hat{\alpha}_p$$

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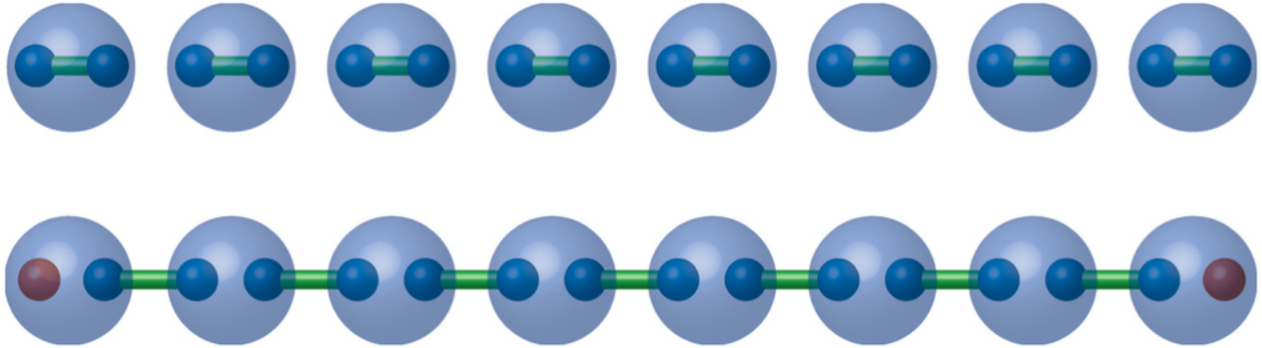

Parafermion

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$$\hat{\alpha}_p\hat{\alpha}_q = \Omega^{\mathrm{sign}(q-p)}\hat{\alpha}_q\hat{\alpha}_p$$

$f = 0 \rightarrow$ parafermions at the edg, $\hat{\alpha}_1$ & $\hat{\alpha}_{2L}$, absent form the Hamiltonian!

The missing **two parafermions** encode an **N-fold degenerate subspace!**

Majoranas vs. parafermions

- Majorana modes can be potentially realized in non-interacting systems. (*i.e.* mean-field description is sufficient)

A.Hutter, D. Loss Phys. Rev. B **93**, 125105 (2016)

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- $\mathbb{Z}_{odd}$ parafermions: route to universality

A.Hutter, D. Loss Phys. Rev. B **93**, 125105 (2016)

Parafermion signatures

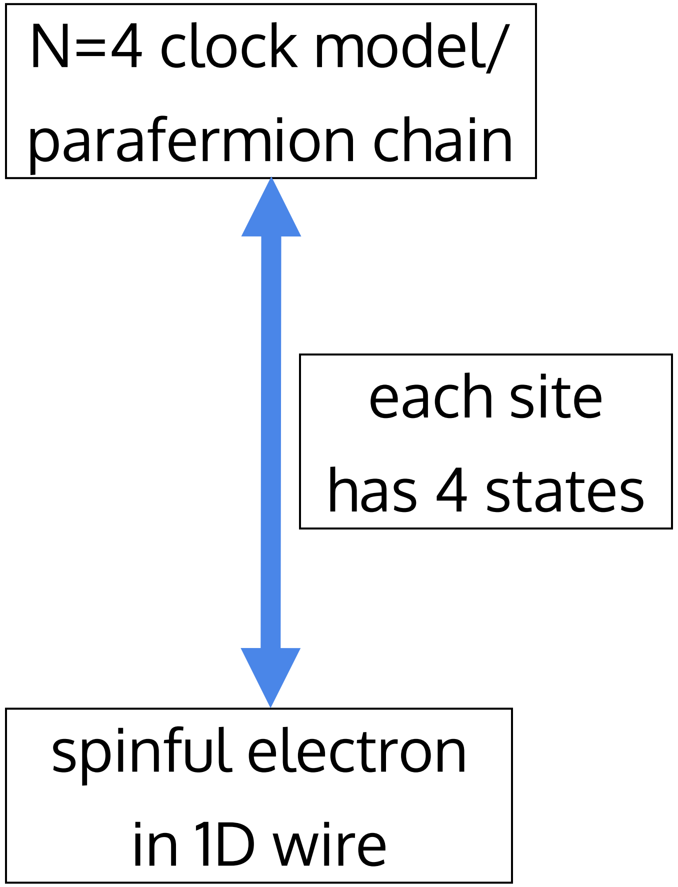
- Robustness against disorder
- Highly (>2) degenerate groundstate
- Localized zero-energy excitations
- Nontrivial (fractional) Josephson effect

$\mathbb{Z}_n \rightarrow 2n\pi$ periodic

J. Alicea, P. Fendley Annu. Rev. Condens. Matter Phys. **7**,119 (2016.)

$\mathbb{Z}_4$ parafermions from ordinary fermions

Hamiltonian in fermion language ...



A. Calzona, T. Meng, M. Sassetti, T. L. Schmidt
Phys. Rev. B **98**, 201110(R) (2018)

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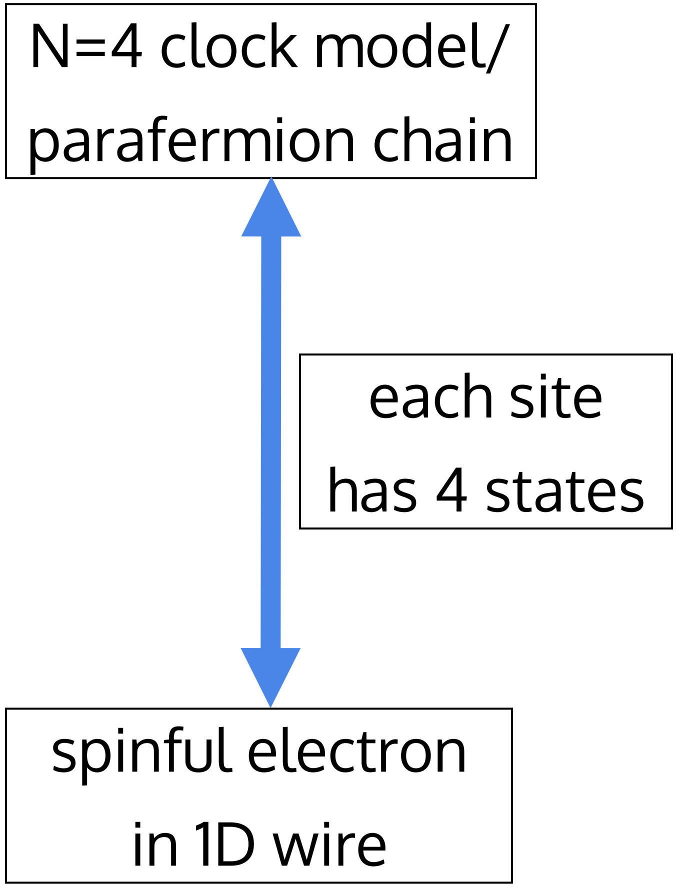

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Hamiltonian in fermion language ...

$$H = H^{(2)} + H^{(4)} + H^{(6)}$$

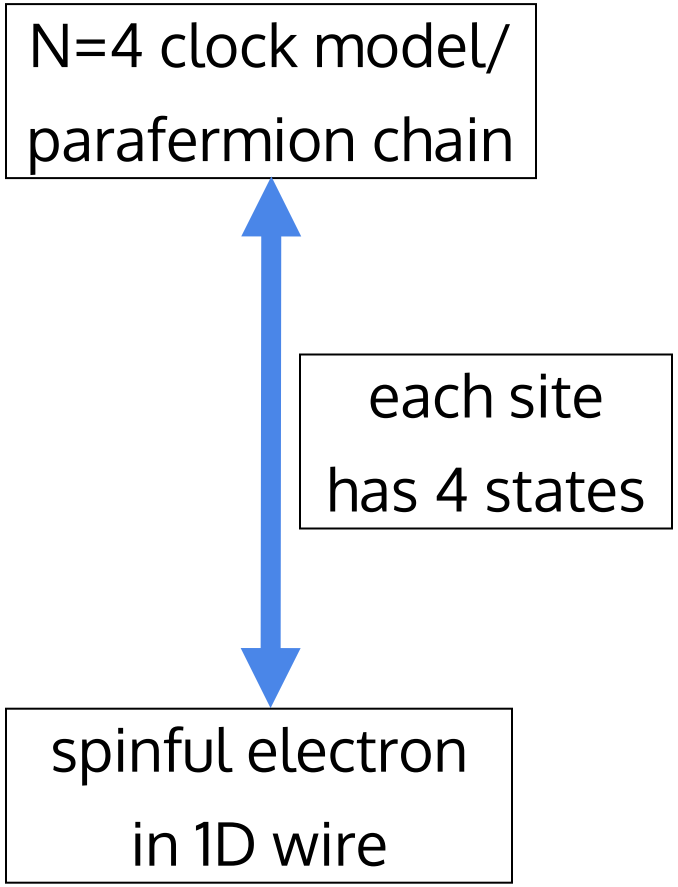
$$H^{(2)} = -J \sum_{\sigma,j} \left[c^\dagger_{\sigma,j} c_{\sigma,j+1} - i\, c^\dagger_{-\sigma,j} c^\dagger_{\sigma,j+1} \right] + h.c.\,,$$

$$\begin{aligned} H^{(4)} = -J \sum_{\sigma,j} \Big[& c^\dagger_{\sigma,j} c_{\sigma,j+1} \left(-n_{-\sigma,j} - n_{-\sigma,j+1}\right) \\ & + c^\dagger_{\sigma,j} c_{-\sigma,j+1}\; i \left(n_{-\sigma,j} + n_{\sigma,j+1}\right) \\ & + c^\dagger_{-\sigma,j} c^\dagger_{\sigma,j+1}\; i \left(n_{\sigma,j} + n_{-\sigma,j+1}\right) \\ & + c^\dagger_{\sigma,j} c^\dagger_{\sigma,j+1} \left(n_{-\sigma,j} - n_{-\sigma,j+1}\right) \Big] + h.c.\,, \end{aligned}$$

$$\begin{aligned} H^{(6)} = -J \sum_{j} \Big[& -2i\, c^\dagger_{\sigma,j} c_{-\sigma,j+1} \left(n_{-\sigma,j} n_{\sigma,j+1}\right) \\ & -2i\, c^\dagger_{-\sigma,j} c^\dagger_{\sigma,j+1} \left(n_{\sigma,j} n_{-\sigma,j+1}\right) \Big] + h.c. \end{aligned}$$

... is complicated with 3 body interactions encoded in the $H^{(6)}$ term

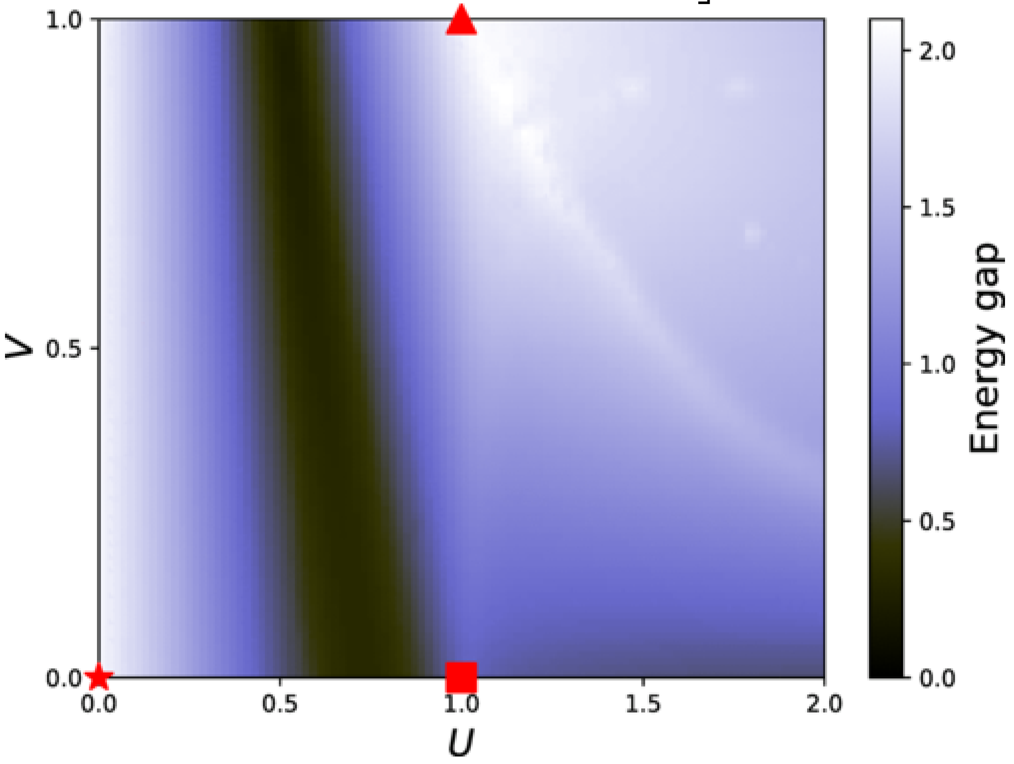
$\mathbb{Z}_4$ parafermions from ordinary fermions



Hamiltonian in fermion language ...

$$\bar{H}(U,V) = H^{(2)} + U\left[V\left(H^{(4)} + H^{(6)}\right) + (1-V)\bar{H}^{(4)}\right]$$

$$\bar{H}^{(4)} = -J\sum_{\sigma,j}\Big[c^\dagger_{\sigma,j}c_{\sigma,j+1}\left(-n_{-\sigma,j} - n_{-\sigma,j+1}\right) + c^\dagger_{\sigma,j}c^\dagger_{\sigma,j+1}\left(n_{-\sigma,j} - n_{-\sigma,j+1}\right)\Big] + h.c.$$



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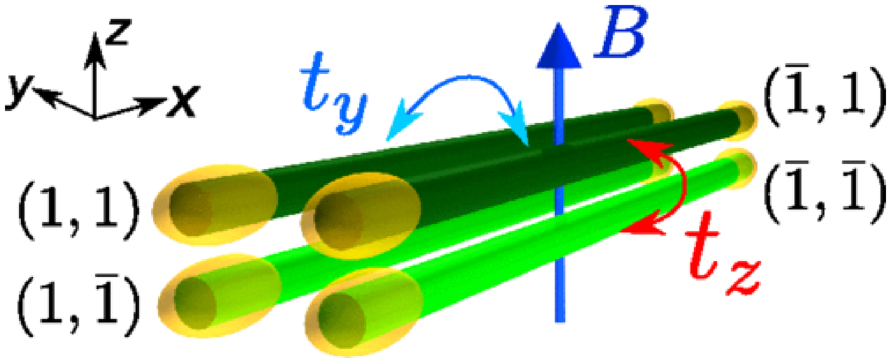
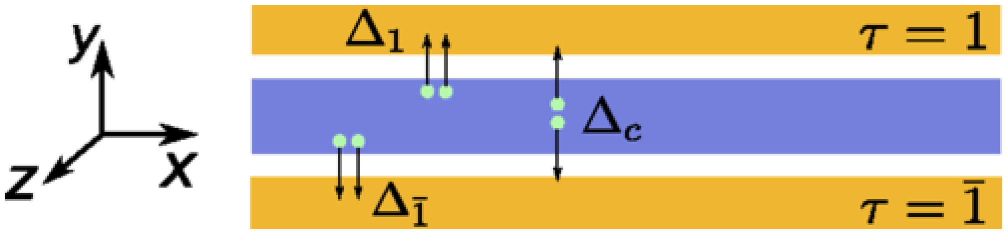
Possible experimental blueprints





J. Klinovaja and D. Loss

Phys. Rev. Lett. **112**, 246403 (2014)

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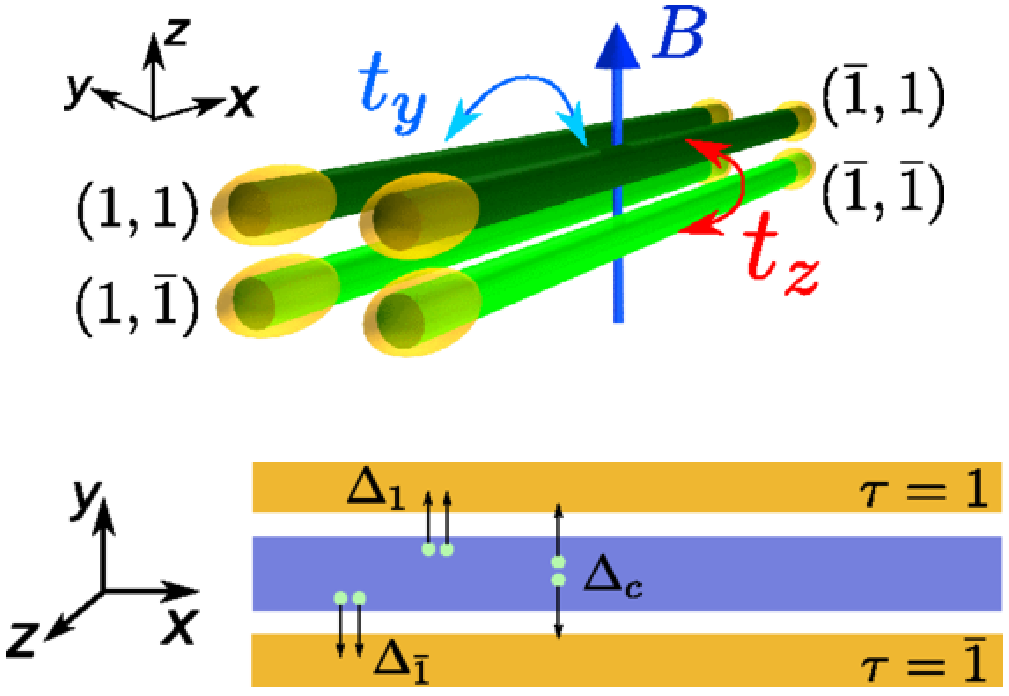

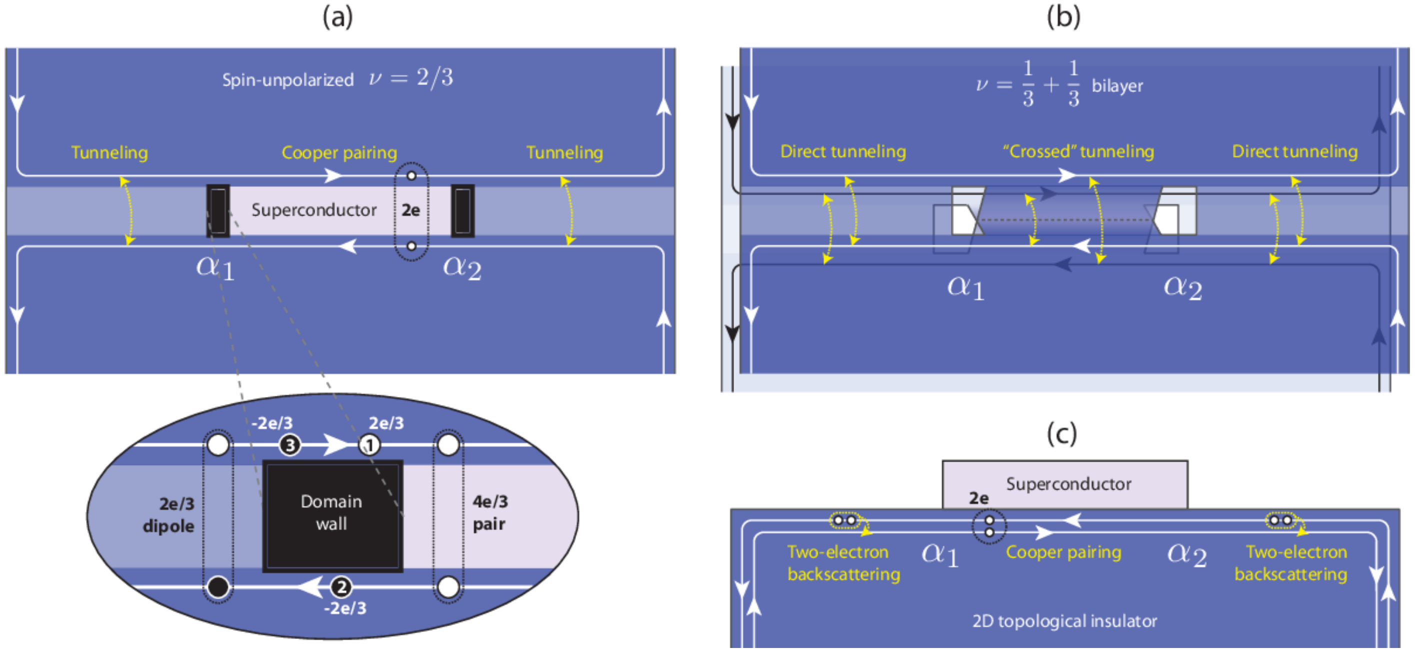

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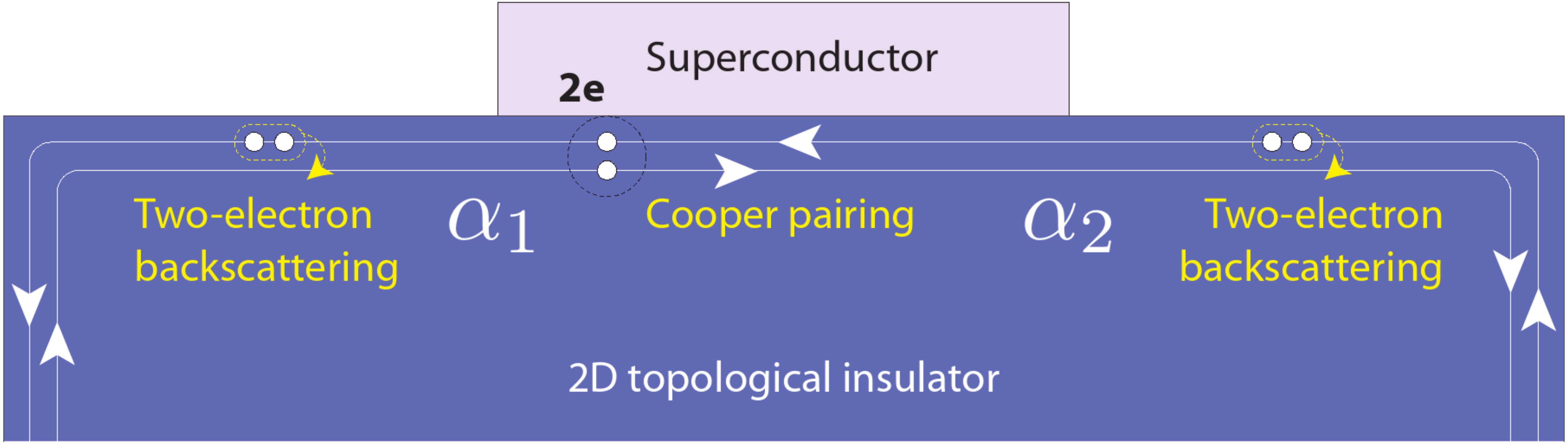
Parafermions at TI edge



F. Zhang, C. L. Kane, Phys. Rev. Lett., **113**, 036401 (2014).
C. P. Orth *et al.* Phys. Rev. B, **91**, 081406 (2015).
J. Alicea, P. Fendley Annu. Rev. Condens. Matter Phys. **7**,119 (2016.)

bosonised models

goal: microscopic model + DMRG

ITensor
Budapest DMRG

A model for 2DTI that can be digested by DMRG?

The model

$$H = H_k + H_{sc} + H_{int}$$

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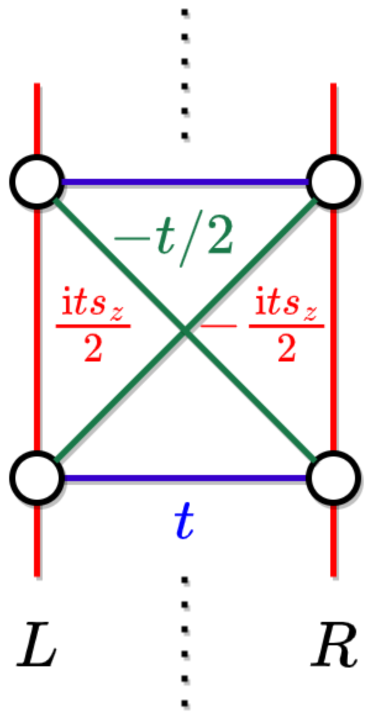
$$H_k = \sum_{m\sigma} \begin{pmatrix} c^\dagger_{mL\sigma} & c^\dagger_{mR\sigma} \end{pmatrix} \begin{pmatrix} -\mu & t \\ t & -\mu \end{pmatrix} \begin{pmatrix} c_{mL\sigma} \\ c_{mR\sigma} \end{pmatrix}$$
$$-\frac{t}{2} \sum_{m\sigma} \left[\begin{pmatrix} c^\dagger_{m+1L\sigma} & c^\dagger_{m+1R\sigma} \end{pmatrix} \begin{pmatrix} \mathrm{i}\sigma & 1 \\ 1 & -\mathrm{i}\sigma \end{pmatrix} \begin{pmatrix} c_{mL\sigma} \\ c_{mR\sigma} \end{pmatrix} + \text{h.c.} \right]$$



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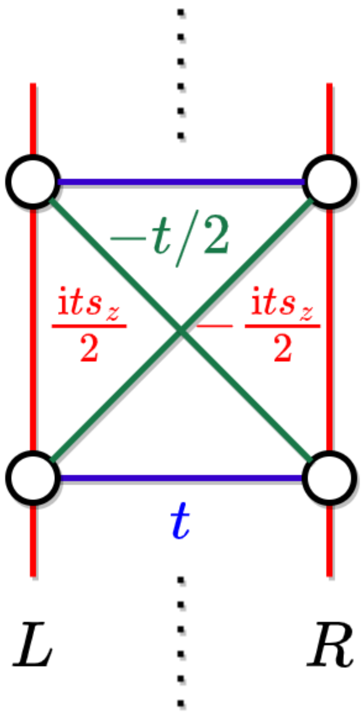

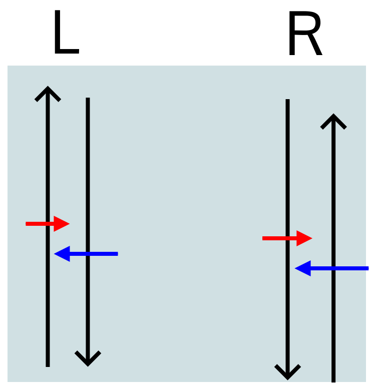

$$H_k \approx p s_z \zeta_z$$

- two "disconnected edges" $\zeta = L, R$

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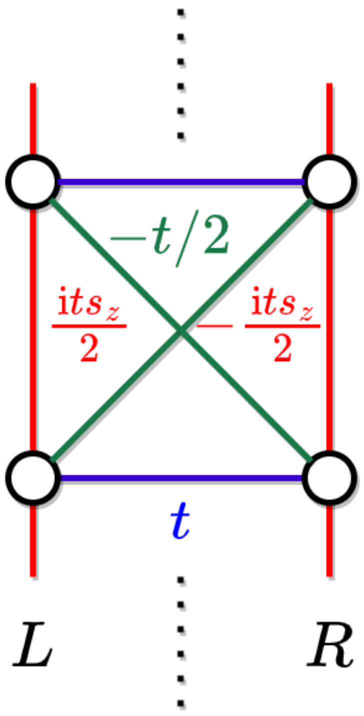

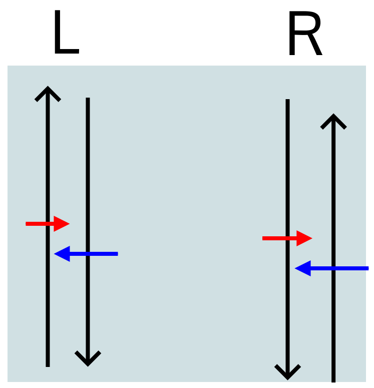

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- two "disconnected edges" $\zeta = L, R$
- explicit superconductivity and interactions

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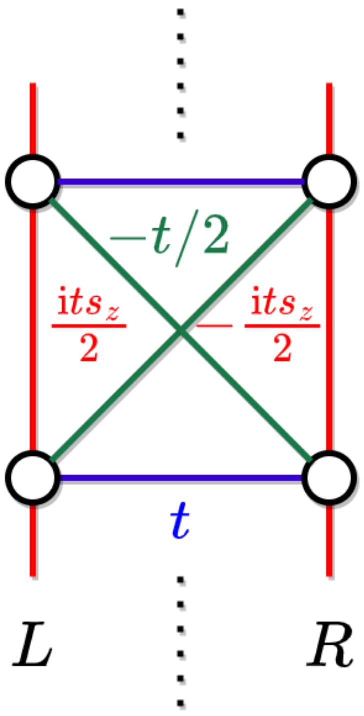

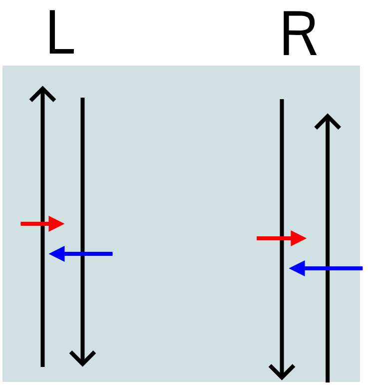

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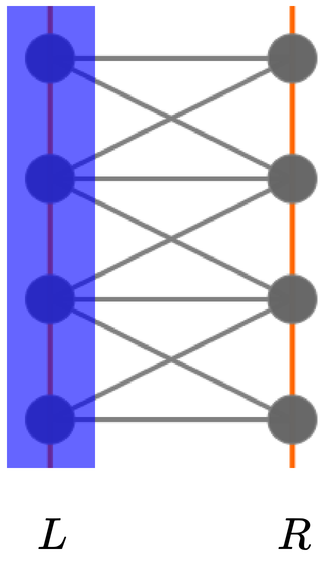
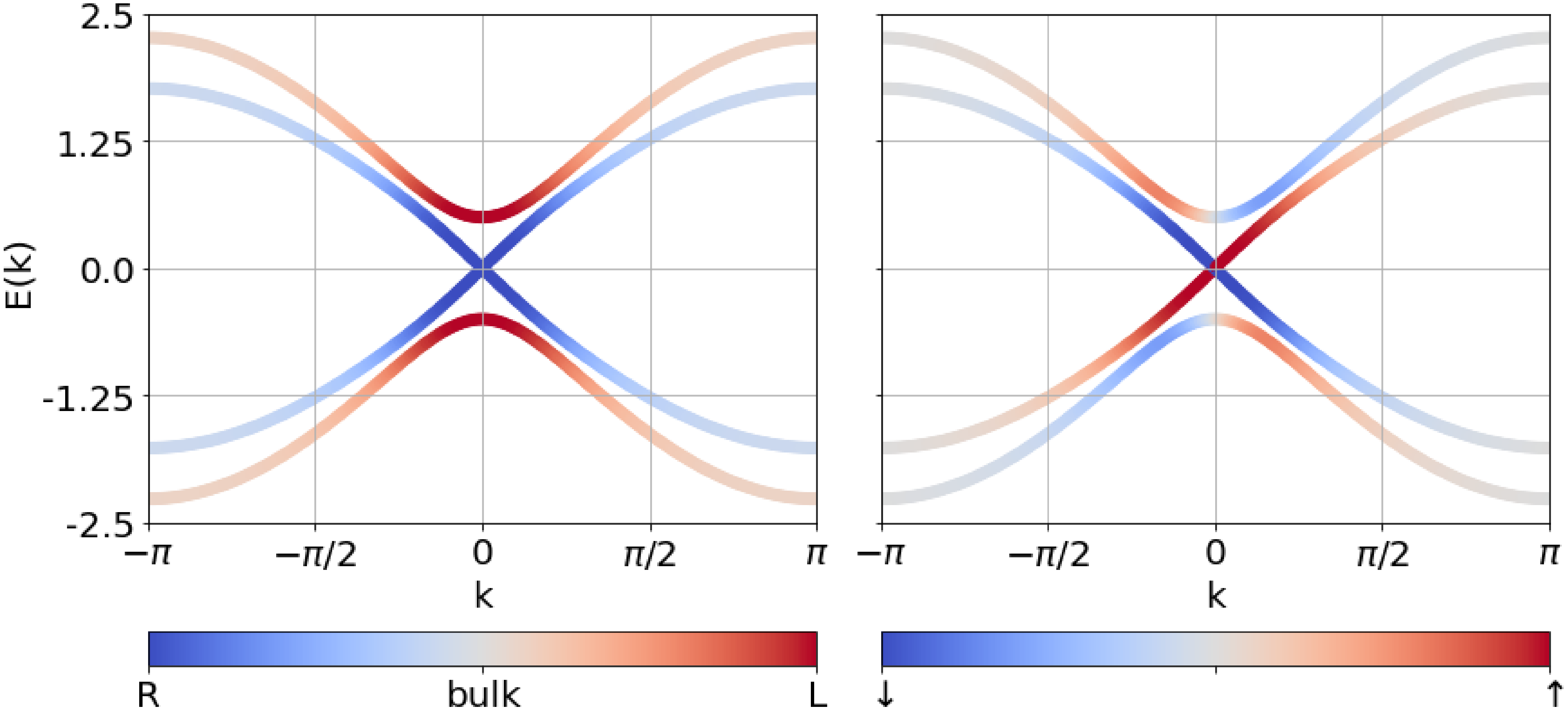
- two "disconnected edges" $\zeta = L, R$
- explicit superconductivity and interactions
- time reversal symmetry

Single particle spectrum

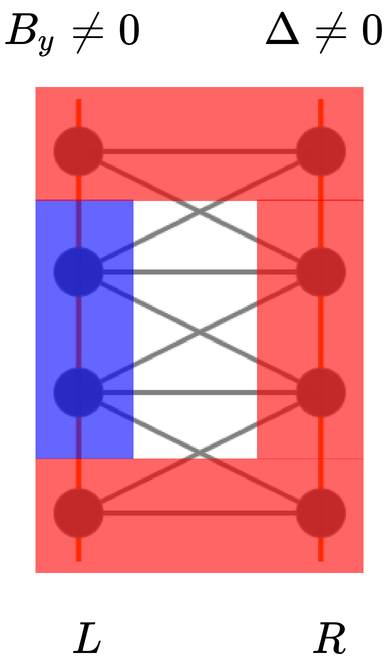
$B_y \neq 0$



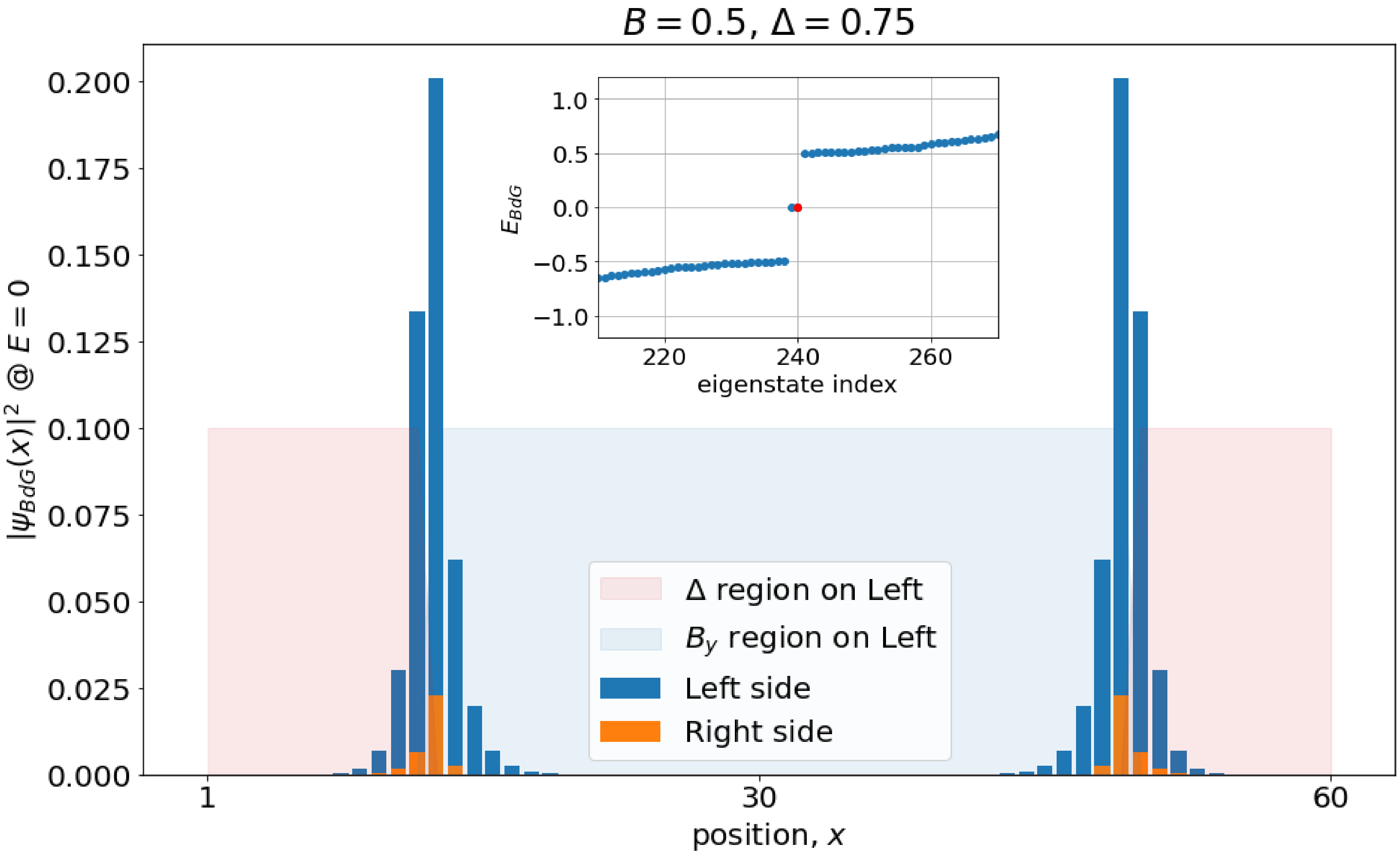


small B_y on the left for better visibility

We still have Majoranas !

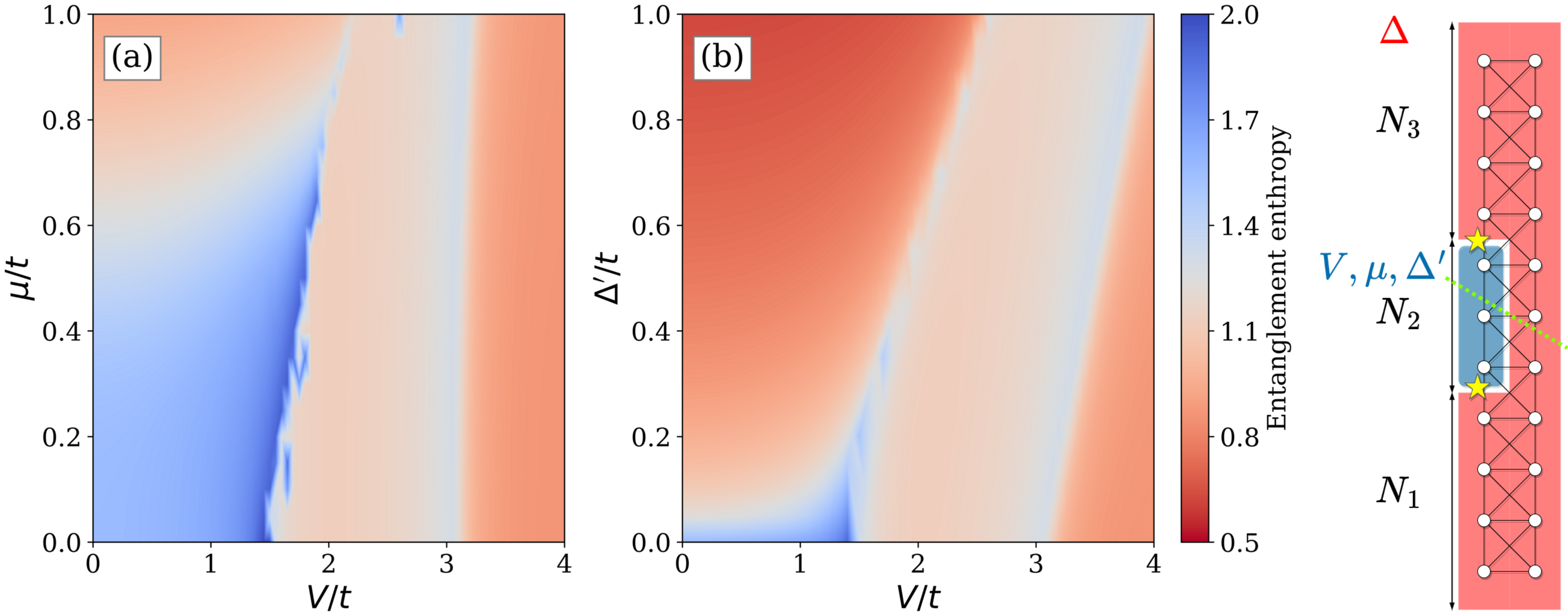


$$V = 0$$

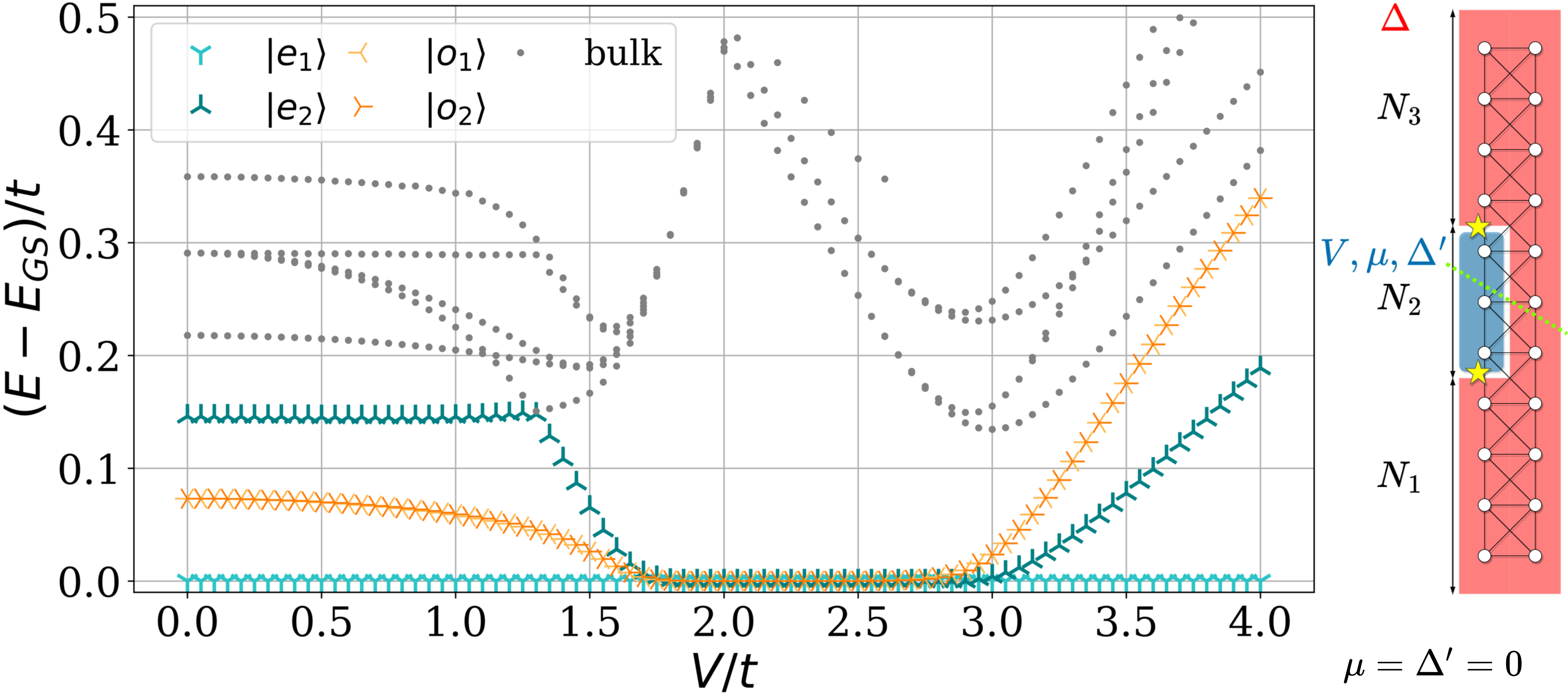


Fu, Kane Phys. Rev. Lett. **100**, 096407 (2008)

Finite-size DMRG calculations: phase diagram

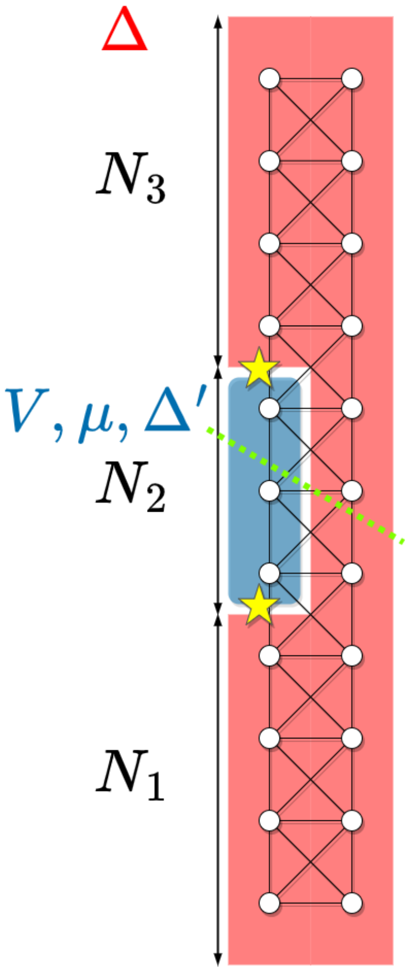


Finite-size DMRG calculations: excitation spectrum



Finite-size DMRG calculations: local quantities

$\mu = \Delta' = 0, V/t = 2.2$



$\langle GS_p | n_i | GS_q \rangle \propto \delta_{pq}$

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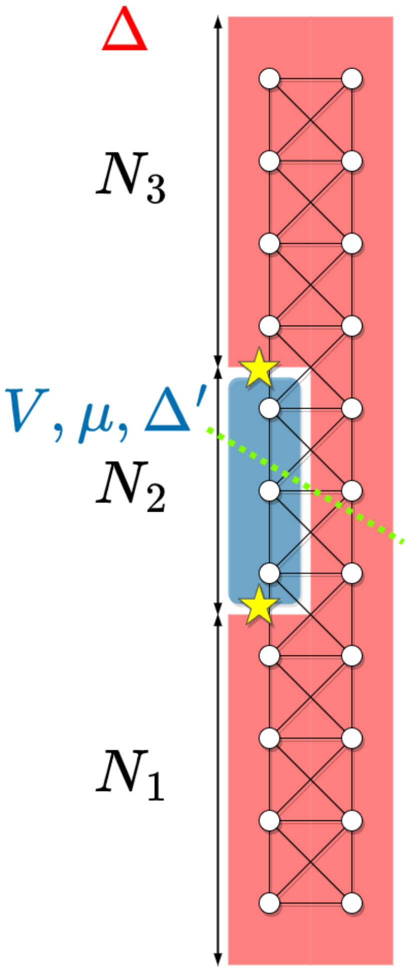

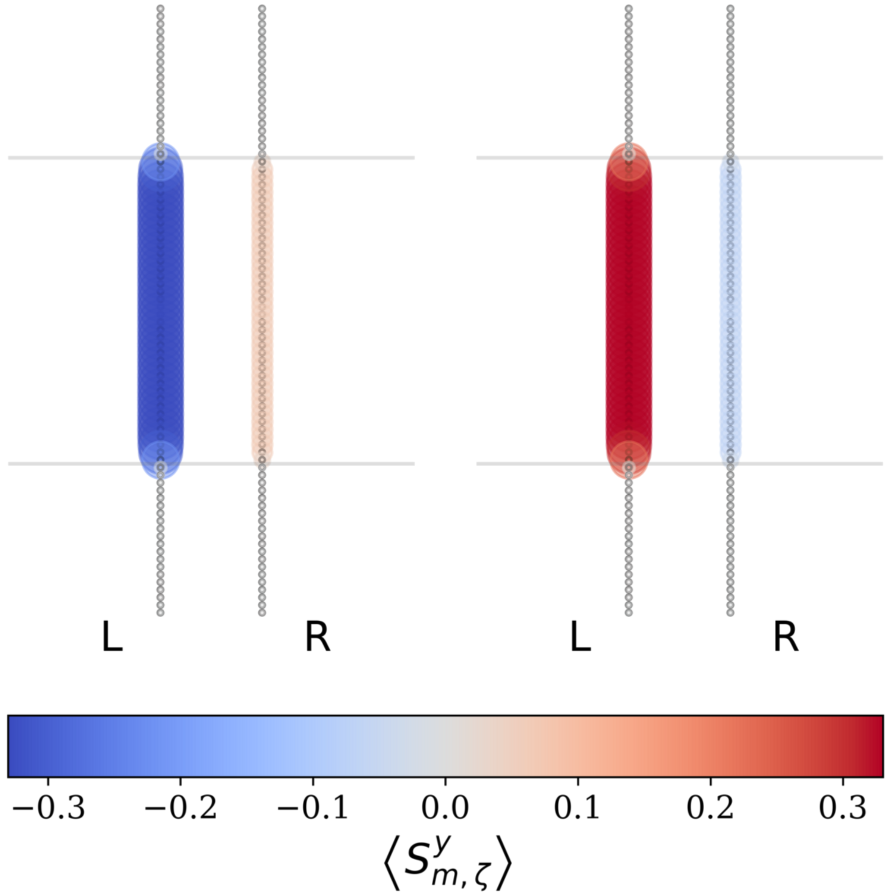

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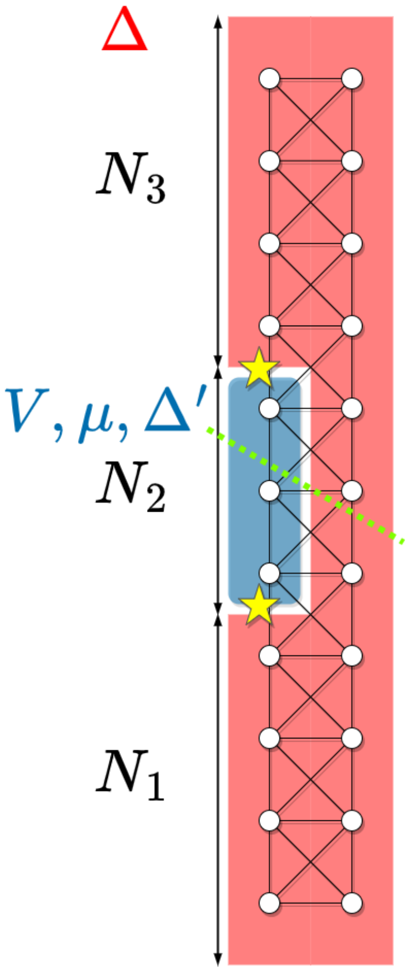

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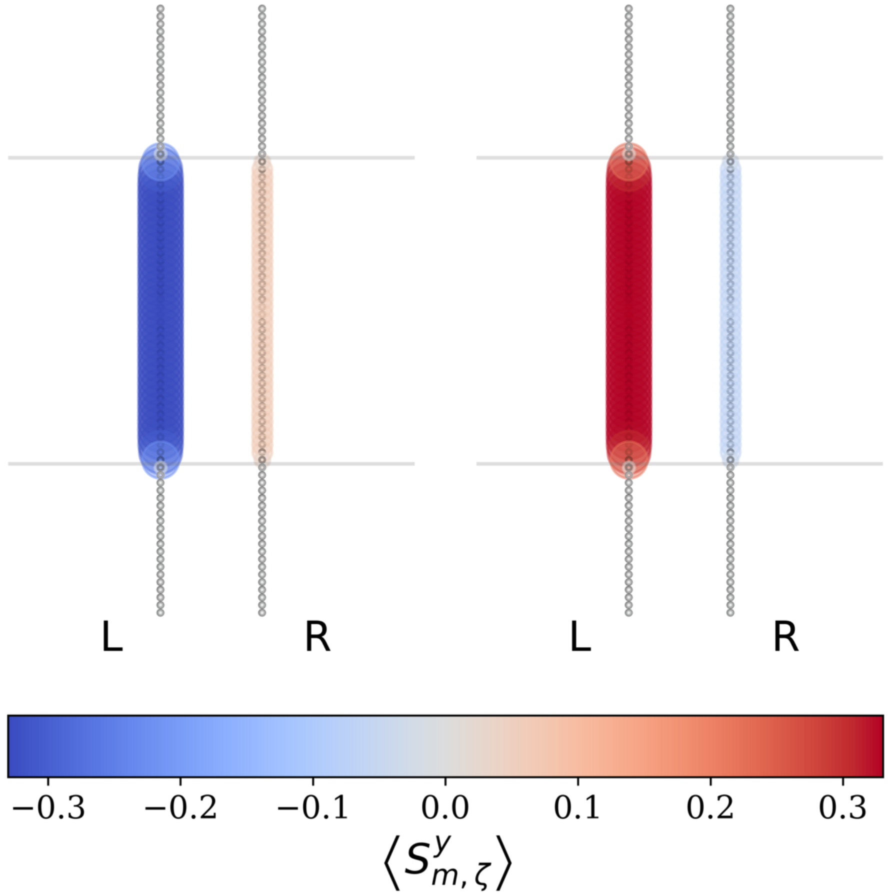

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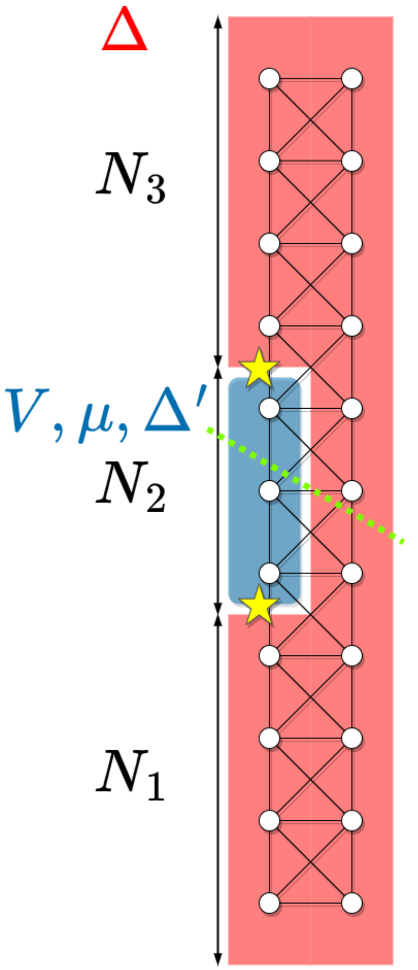
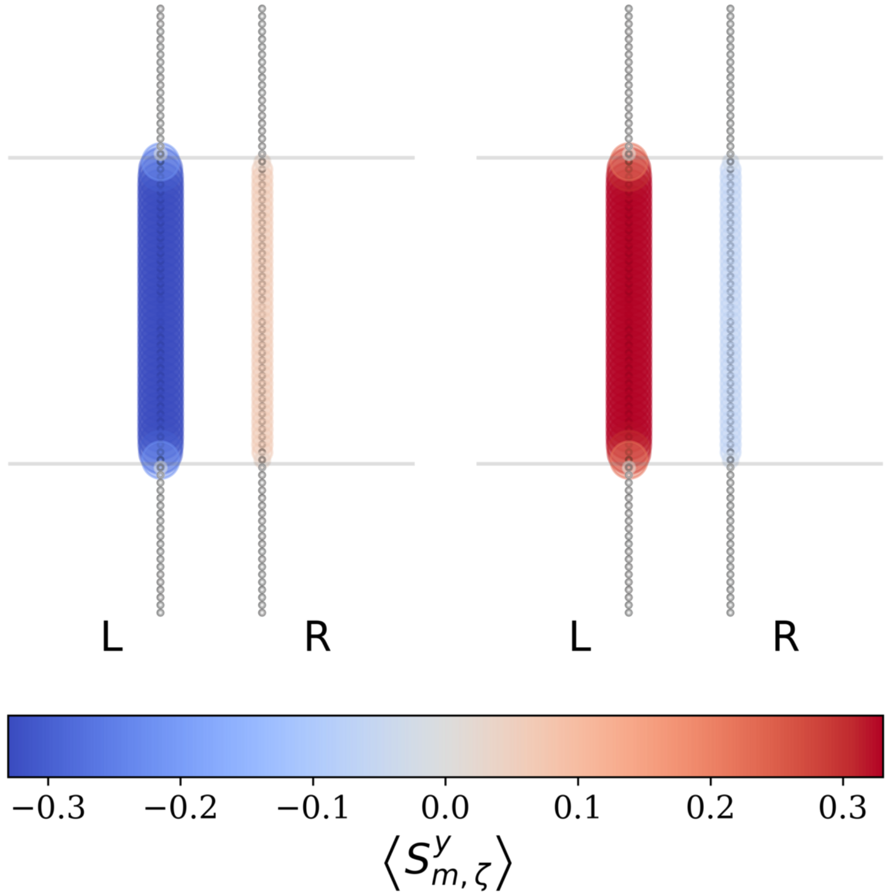
$$= \sum_{m,\zeta} \frac{V_{m,\zeta}}{2} \left[S^x_{m,\zeta} S^x_{m+1,\zeta} - S^y_{m,\zeta} S^y_{m+1,\zeta} \right]$$

Finite-size DMRG calculations: local quantities

$\mu = \Delta' = 0, V/t = 2.2$

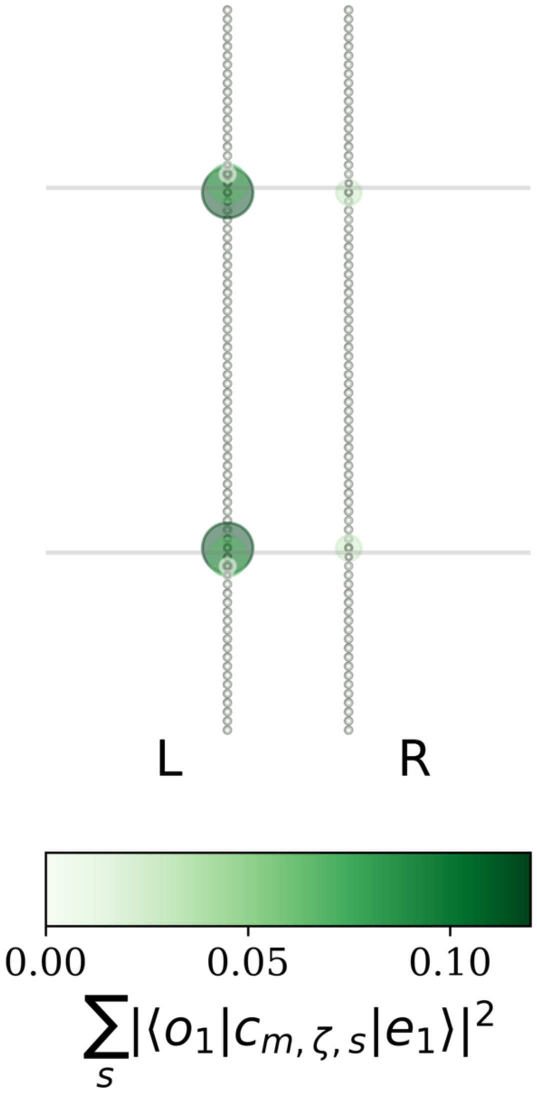


$\langle GS_p | n_i | GS_q \rangle \propto \delta_{pq}$

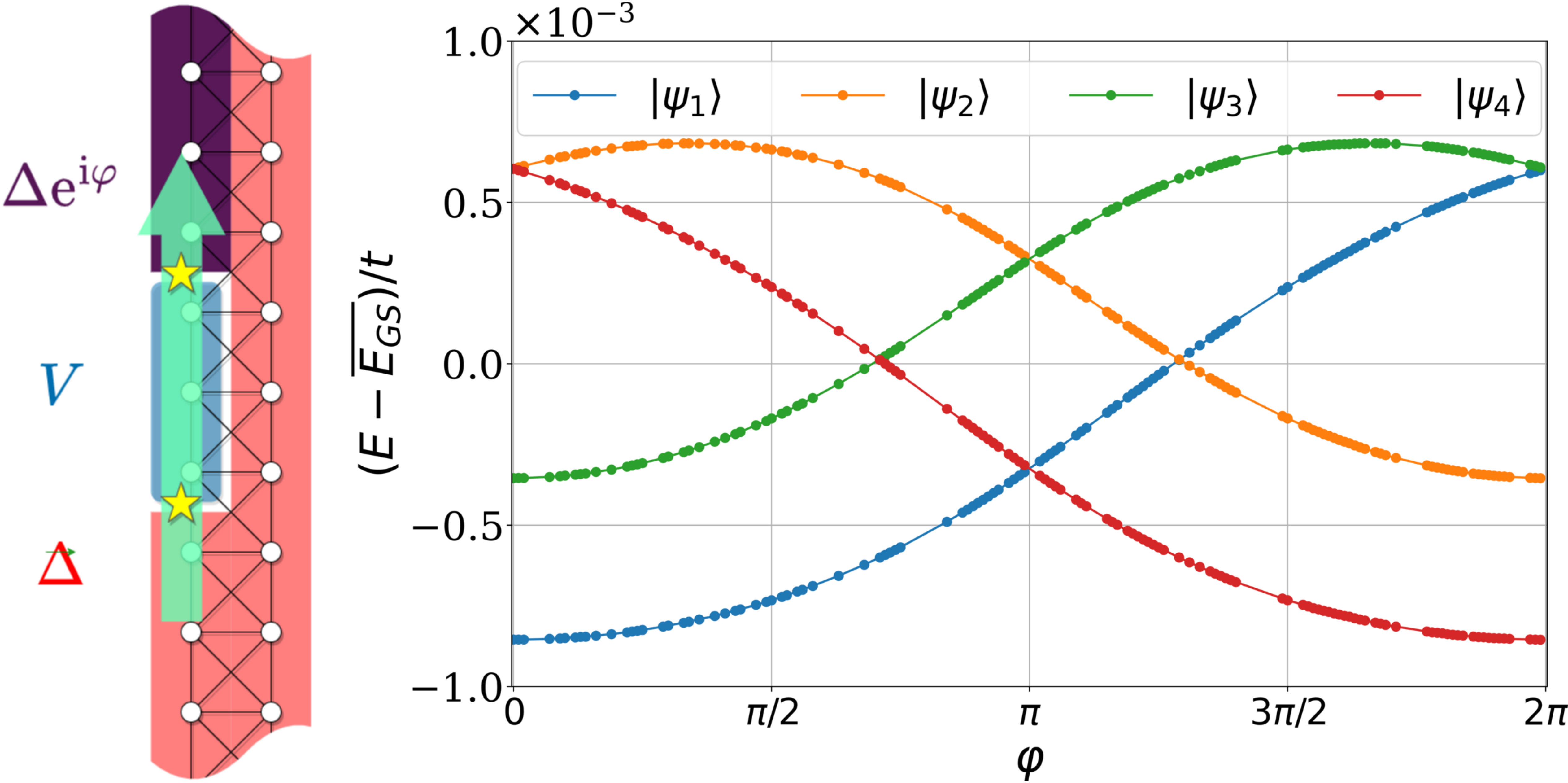


$$H_{int} = \sum_{m\zeta} V_{m\zeta} \left[c^\dagger_{m\zeta\uparrow} c_{m\zeta\downarrow} c^\dagger_{(m+1)\zeta\uparrow} c_{(m+1)\zeta\downarrow} + \text{h.c.} \right]$$

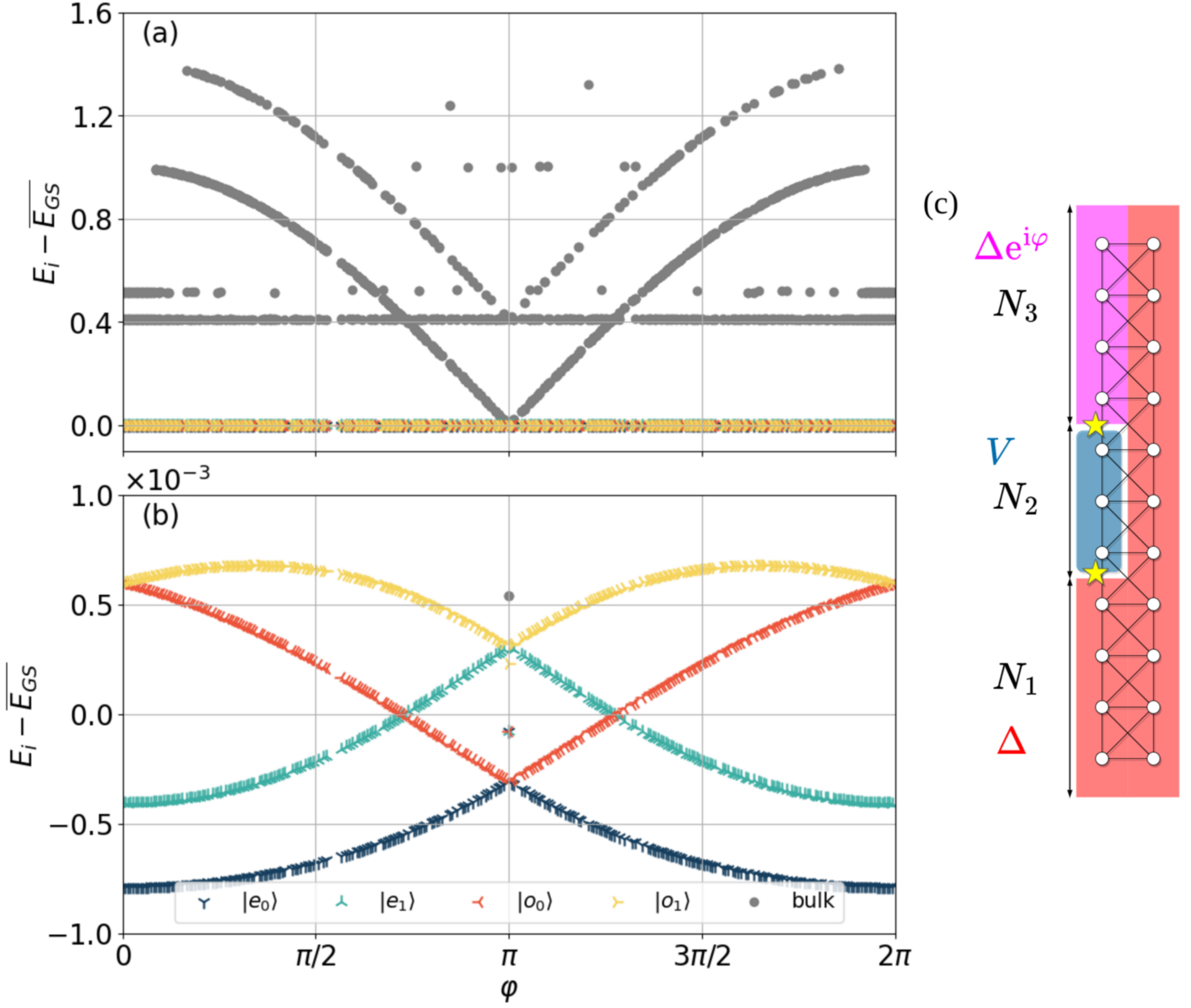
$$= \sum_{m,\zeta} \frac{V_{m,\zeta}}{2} \left[S^x_{m,\zeta} S^x_{m+1,\zeta} - S^y_{m,\zeta} S^y_{m+1,\zeta} \right]$$



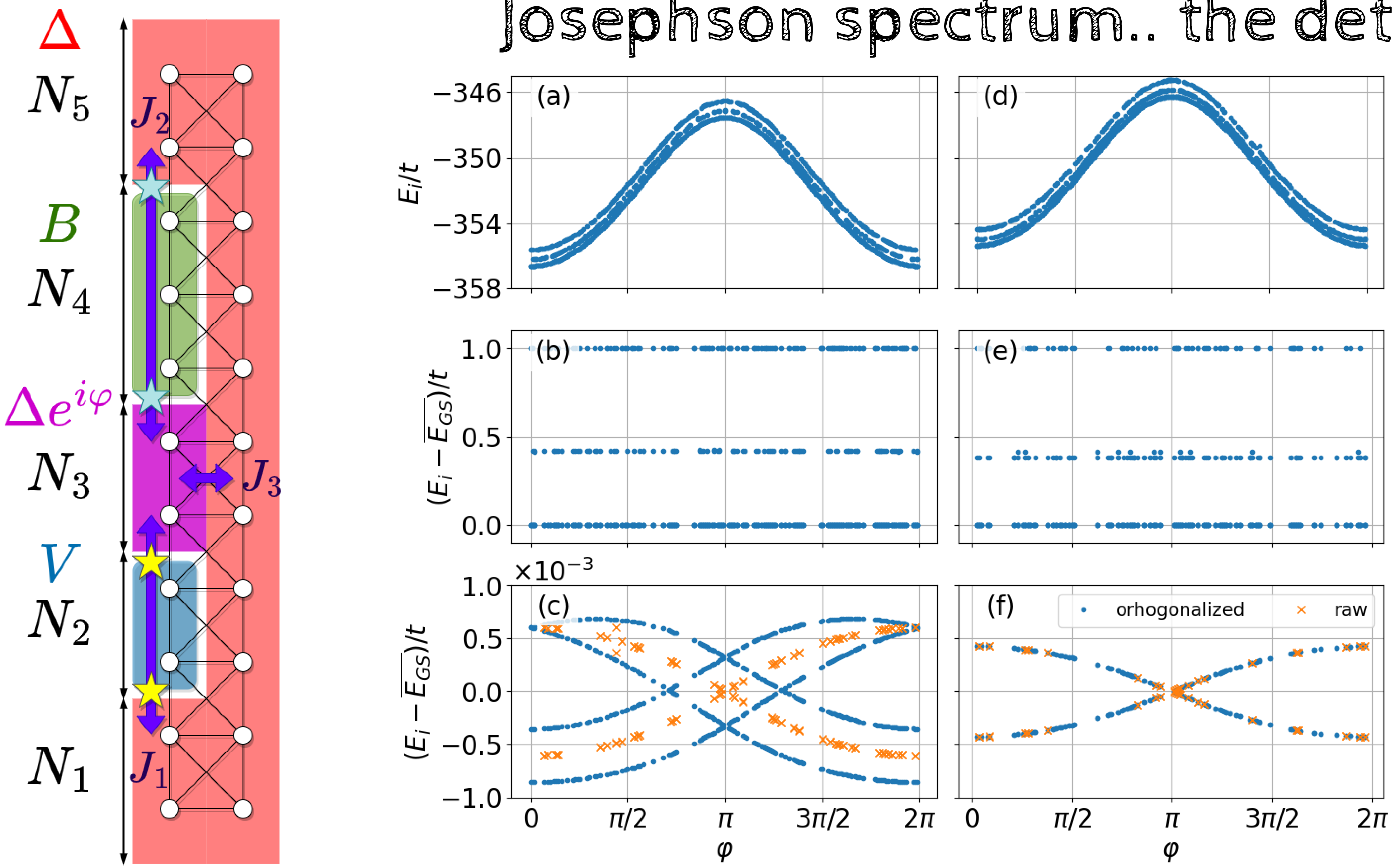
Finite-size DMRG calculations: Josephson spectrum



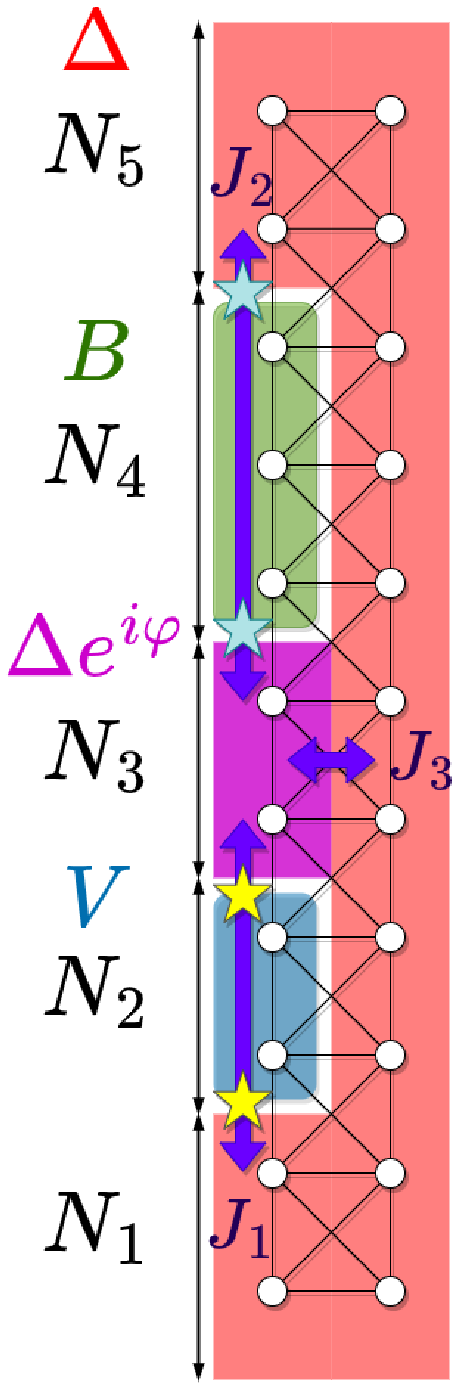
Josephson spectrum.. the details



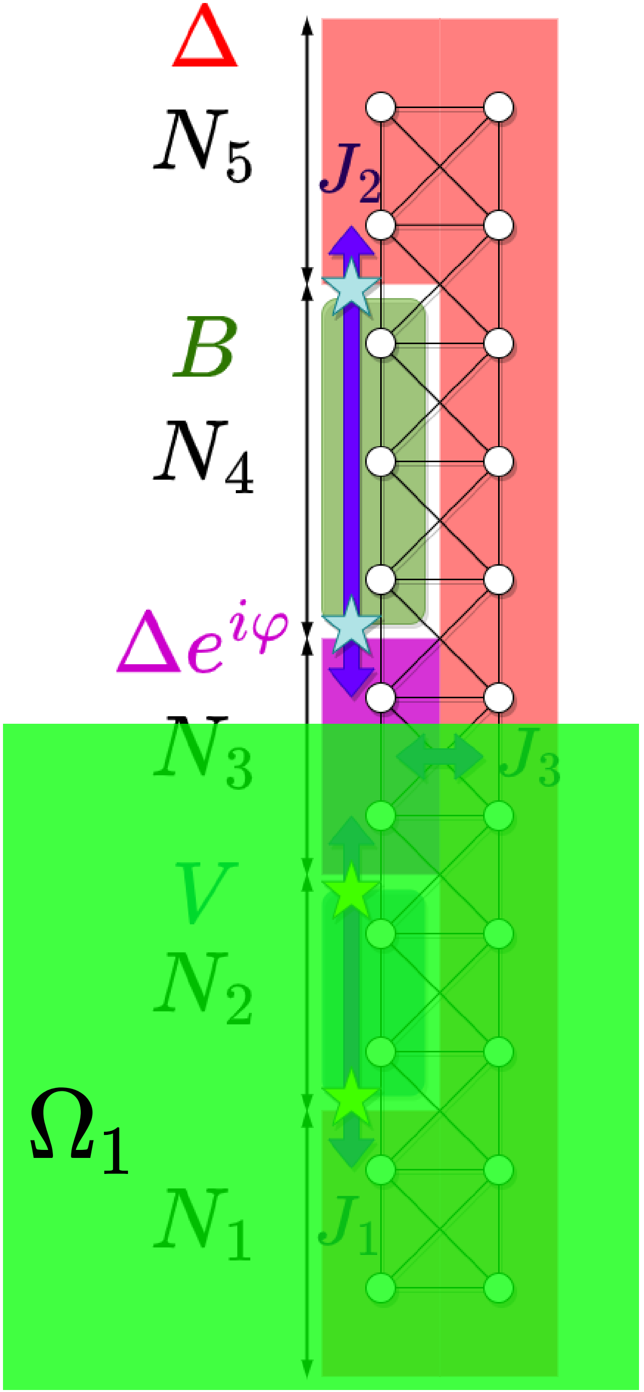
Josephson spectrum.. the details



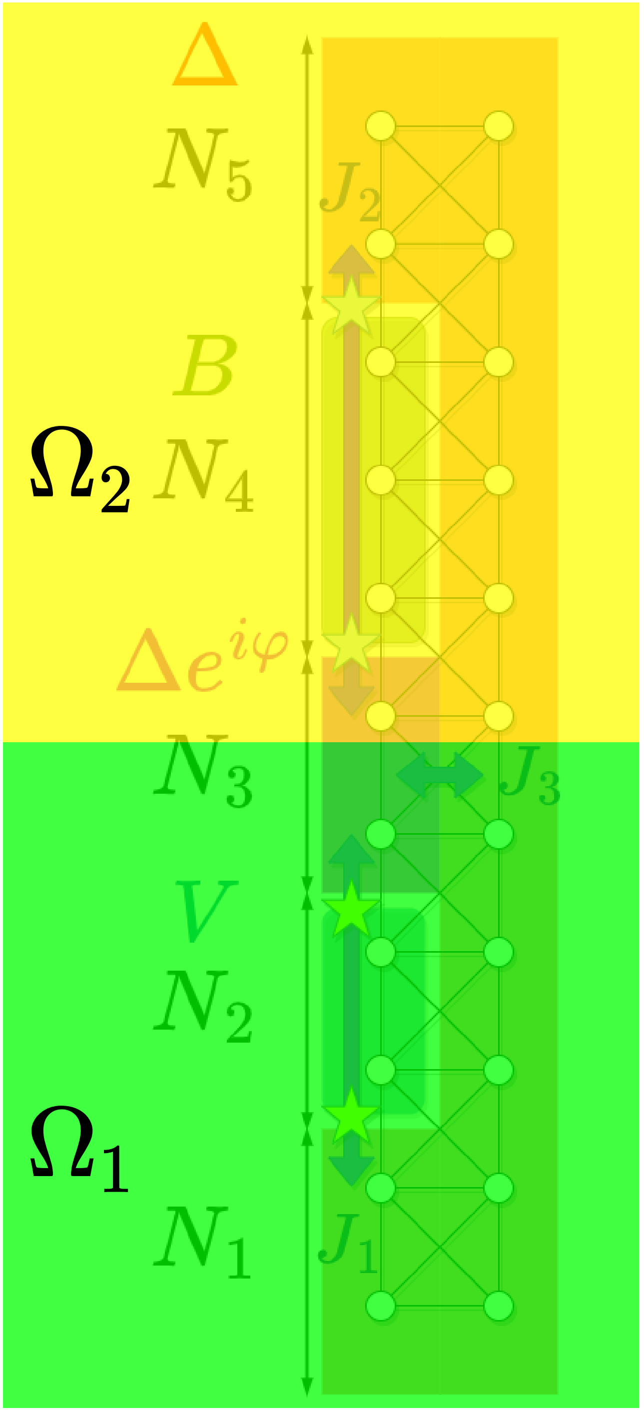
Local parity



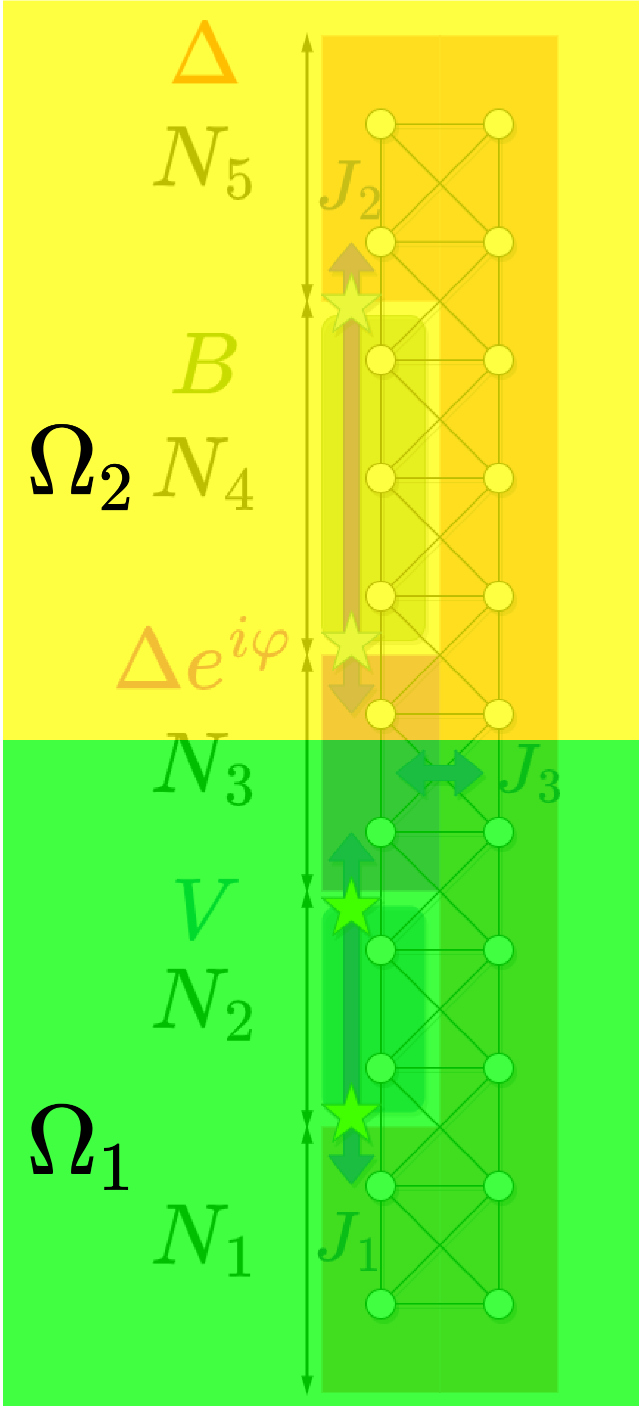
Local parity



Local parity

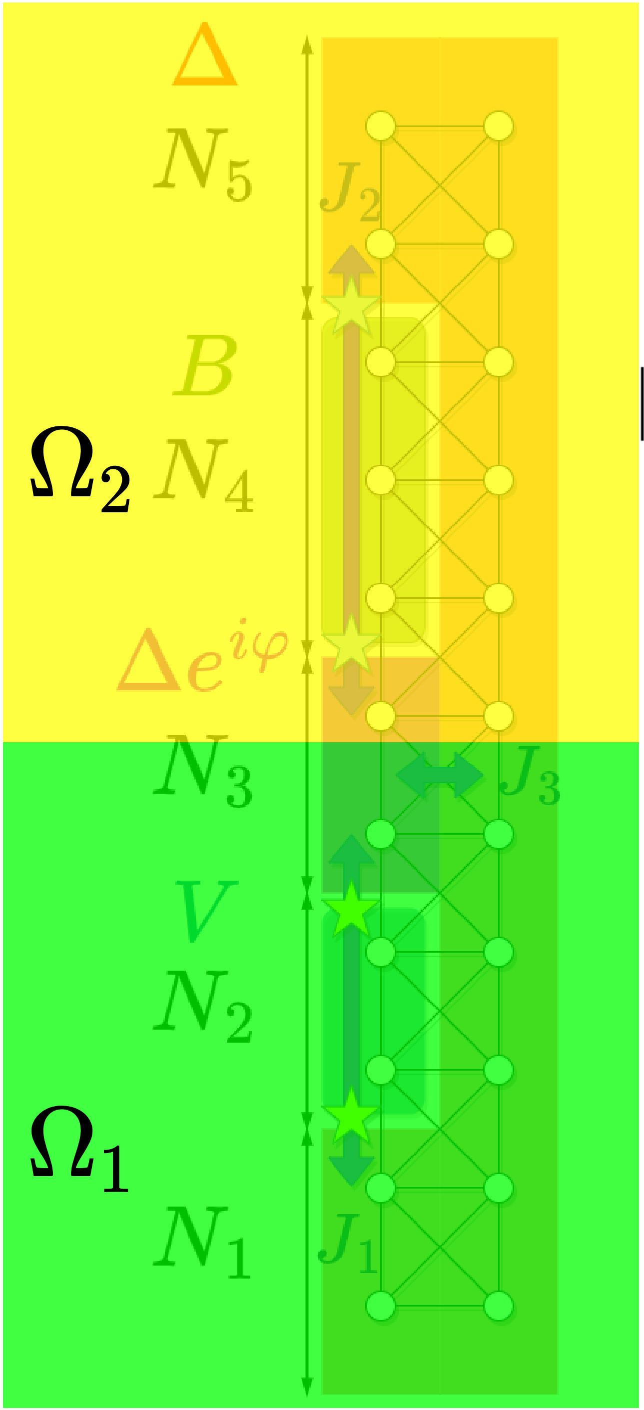


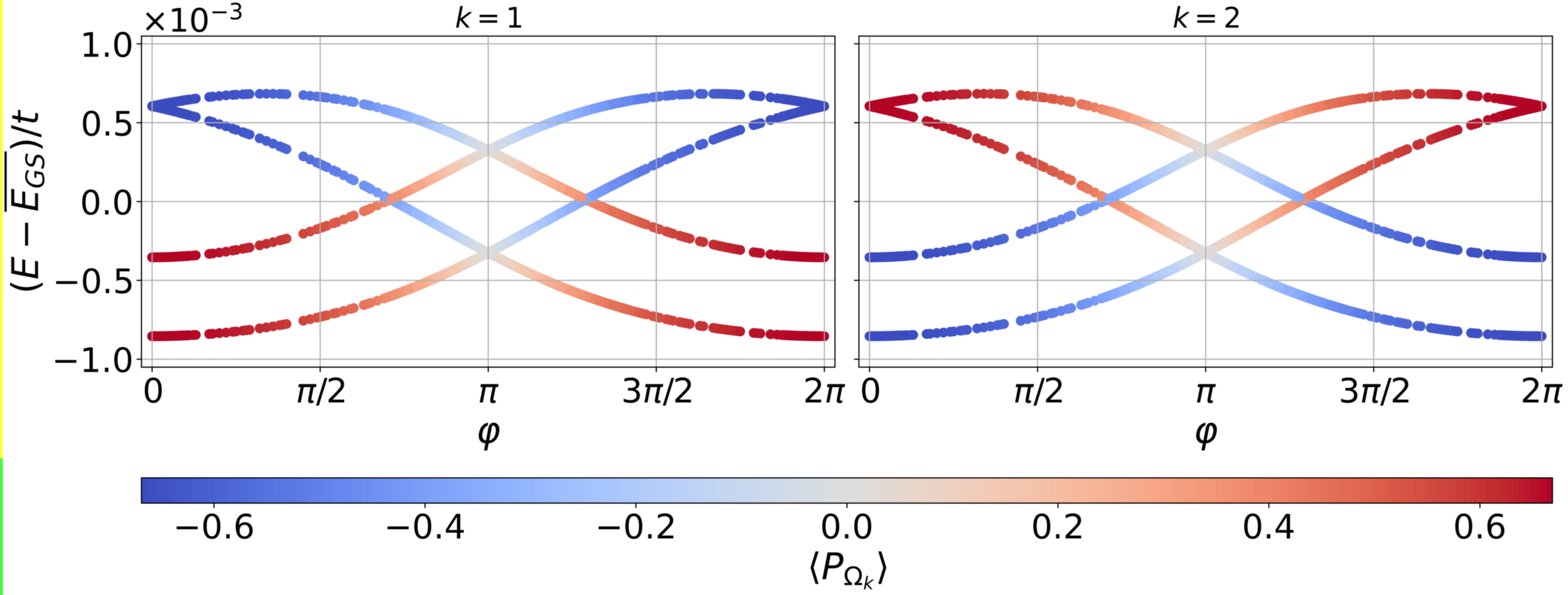
Local parity



$$P_\Omega = \prod_{p\in\Omega,\sigma} (-1)^{n_{p,\sigma}}$$

Local parity





$$P_\Omega = \prod_{p\in\Omega,\sigma} (-1)^{n_{p,\sigma}}$$

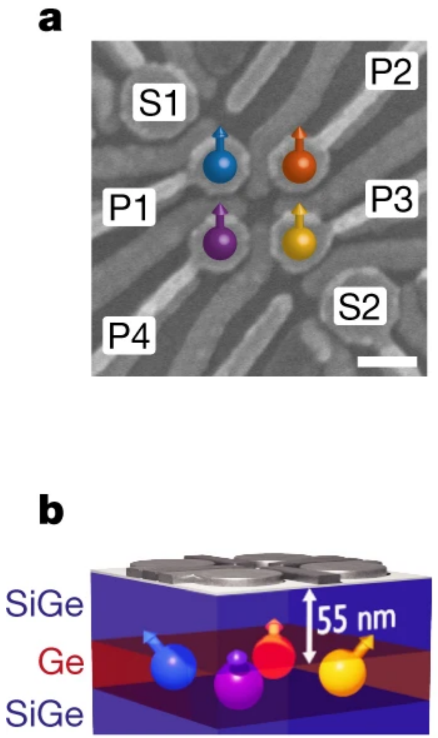
Parafermion signatures

- Robustness against disorder ✓
- Fourfold degenerate groundstate ✓
- Localized zero-energy excitations ✓
- Nontrivial (fractional) Josephson effect ✓

$\mathbb{Z}_4 \rightarrow 8\pi$ periodic

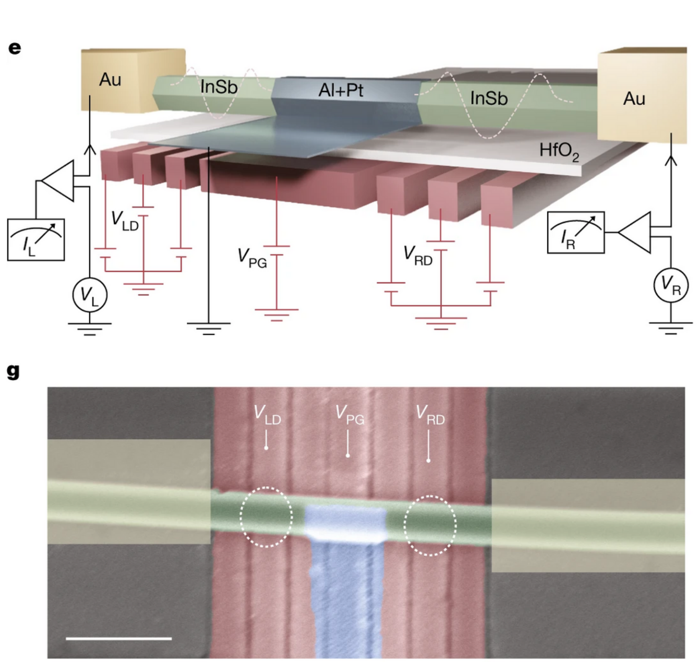
J. Alicea, P. Fendley Annu. Rev. Condens. Matter Phys. **7**,119 (2016.)

Quantum dot arrays for parafermions!!



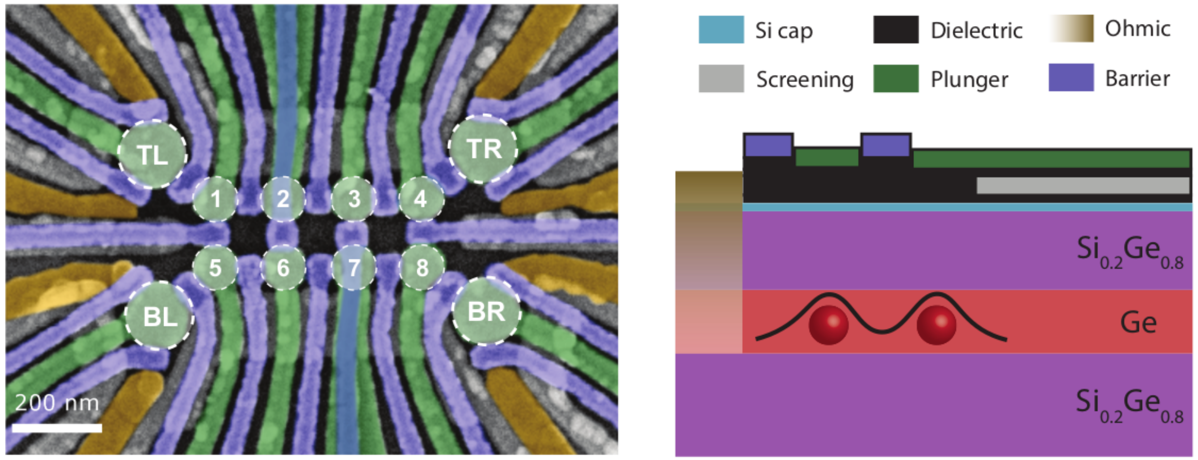
Hendrickx *et al.*

Nature **591**, 580 (2021).

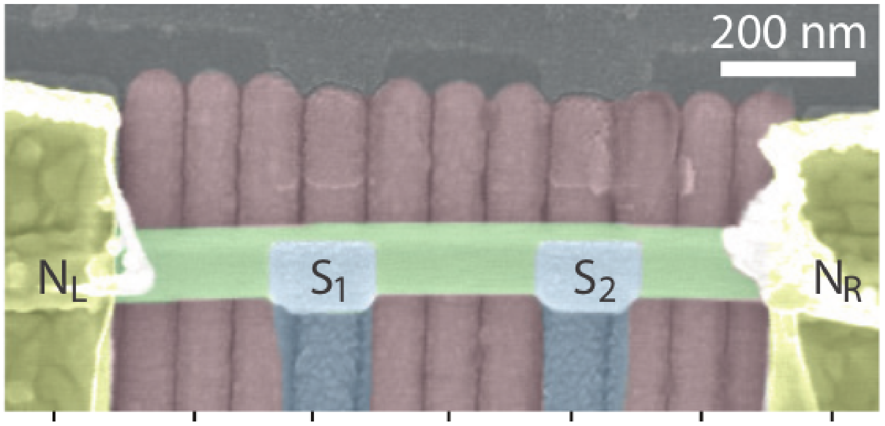


Dvir *et al.*

Nature **614**, 445 (2023)

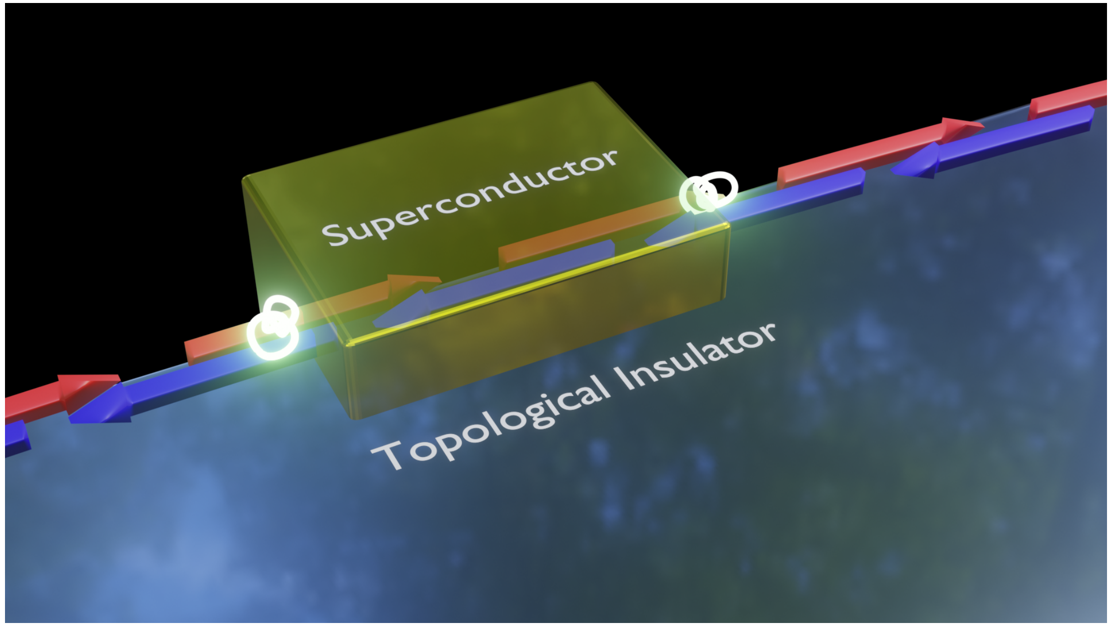


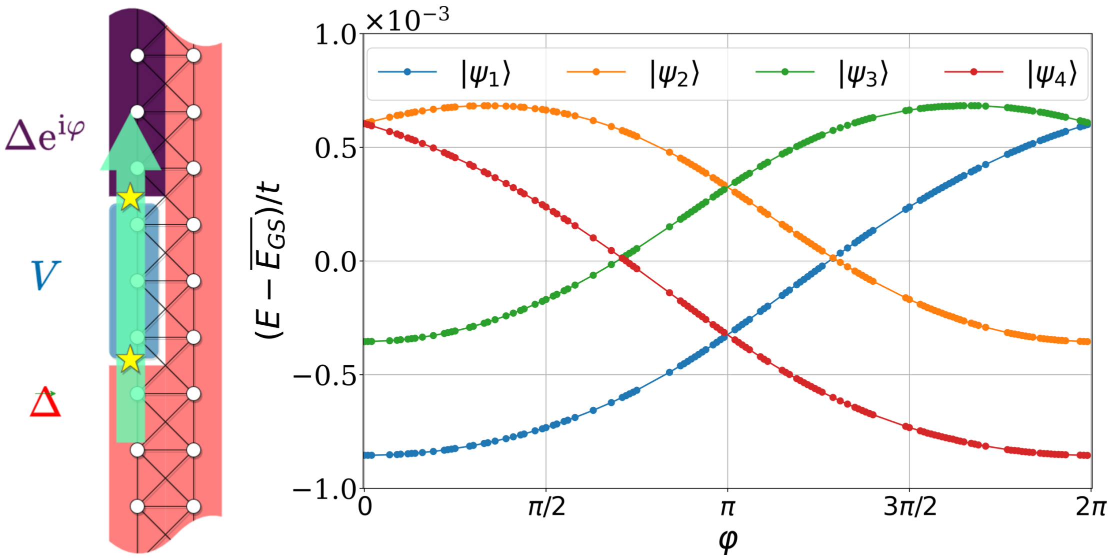
Hsiao *et al.* arXiv:2307.02401 (2023).



Bordin *et al.* arXiv:2306.07696

Summary





- Simple ladder model capable to capture physics at a **single** edge of a TI.
- **Interactions** and **superconductivity** are explicitly taken into account through DMRG calculations.
- **Fourfold degeneracy** and localized interface states can be realized.
- **8π Josephson** spectrum → parafermions!
- New realization avenues for parafermions in **QD arrays** are suggested.

https://arxiv.org/abs/2311.07359