# Entanglement quantification with collective measurements in many-body systems

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Klempt group (Hannover)

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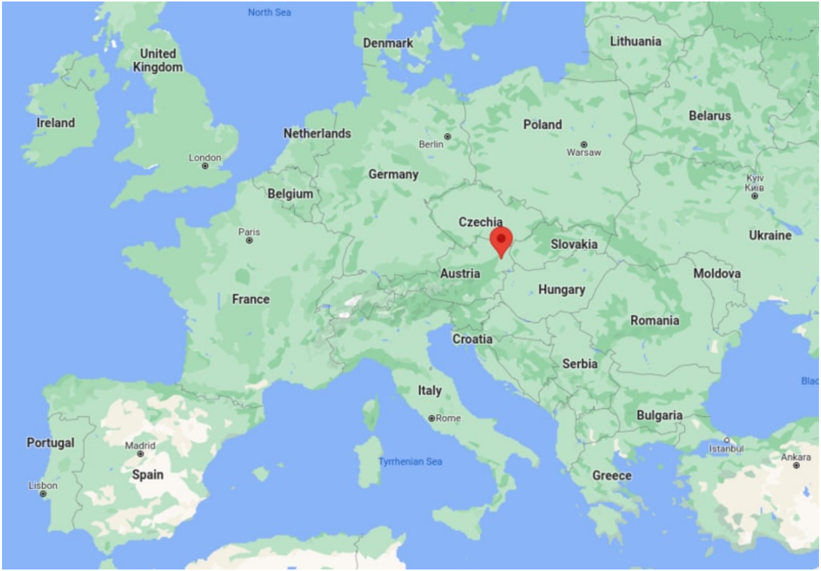
Universidad del Pais Vasco UPV/EHU, May 2025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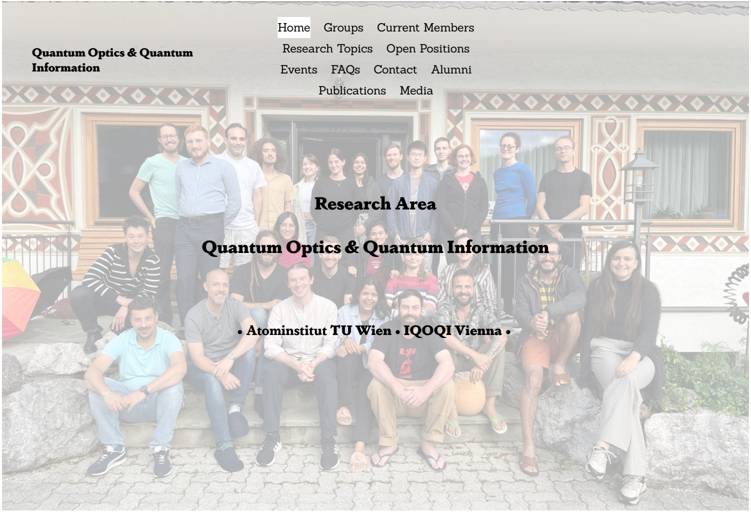
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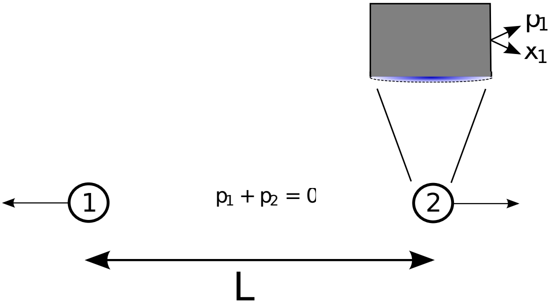
1. Introduction: foundational questions in quantum mechanics

2. Definition of entanglement and basic examples

3. Entanglement detection (witnesses)
   - Bipartite case
   - Multipartite case

4. Entanglement quantification: monotones

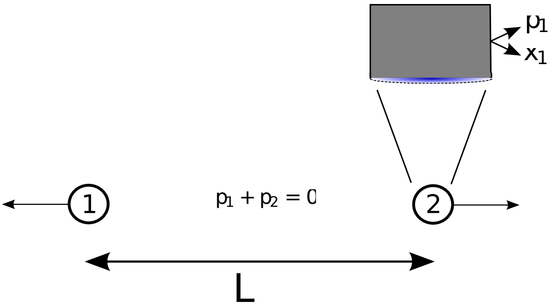
# Einstein, Podolsky, Rosen 1935




$$p_1 + p_2 = 0 \qquad x_1 - x_2 = L$$

definite values cannot be assigned to $(x_1, p_1)$ and $(x_2, p_2)$

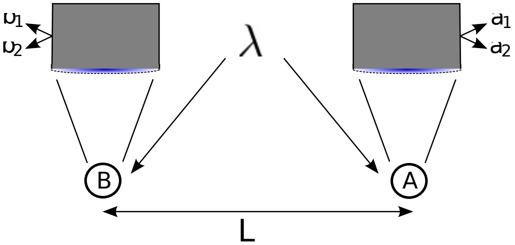
# Einstein, Podolsky, Rosen 1935




$$p_1 + p_2 = 0 \qquad x_1 + x_2 = L$$

The values of $(x_1, p_1)$ and $(x_2, p_2)$ are "entangled", i.e., they are *individually undefined* and *strongly correlated*

# Bell 1964




$$\langle a_1 b_1 \rangle + \langle a_1 b_2 \rangle + \langle a_2 b_1 \rangle - \langle a_2 b_2 \rangle \leq 2$$

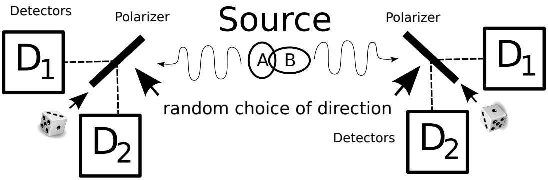
with

$$\langle a_i b_j \rangle = \int x_i y_j \Pr(x_i, y_j)_{a_i, b_j} \mathrm{d}x_i \mathrm{d}y_j$$

and

$$\Pr(x_i, y_j)_{a_i, b_j} = \int \Pr(\lambda) \Pr(x_i | \lambda)_{a_i} \Pr(y_j | \lambda)_{b_j} \mathrm{d}\lambda$$

# Bell 1964




In quantum mechanics:

$$\mathrm{Tr}\,(\rho o) = 2\sqrt{2} > 2$$

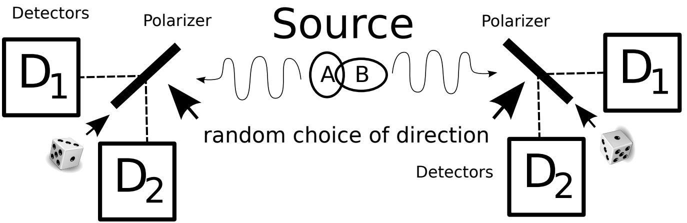
with

$$o = a_1 \otimes b_1 + a_1 \otimes b_2 + a_2 \otimes b_1 - a_2 \otimes b_2$$

$$(a_1, a_2) = (\sigma_z, \sigma_x) \qquad \text{and} \qquad (b_1, b_2) = \left(\frac{\sigma_z + \sigma_x}{\sqrt{2}}, \frac{\sigma_z - \sigma_x}{\sqrt{2}}\right)$$

$$\rho = \frac{\mathbb{1}}{4} + \frac{1}{4} \sum_{k=x,y,z} \sigma_k \otimes \sigma_k$$

# Bell 1964




Experimental test(s): 1972 (Loopholes) - . . . - 2015 (Loophole free)
Local hidden variable theories must be excluded!

Nobel Prize in Physics to Alain Aspect, John F. Clauser and Anton Zeilinger "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science."

Alain Aspect, John F. Clauser and Anton Zeilinger. Credit: Ill. Niklas Elmehed © Nobel Prize Outreach

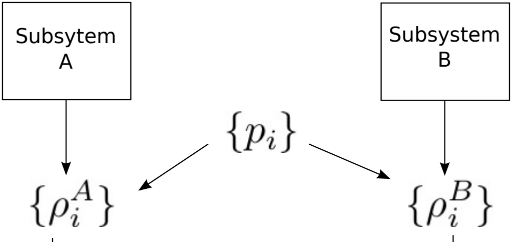
From the press release of Royal Swedish Academy of Sciences:

"Alain Aspect, John F. Clauser and Anton Zeilinger have each conducted groundbreaking experiments using entangled quantum states, where two particles behave like a single unit even when they are separated. Their results have cleared the way for new technology based upon quantum information.

# Can we detect quantum effects at macroscopic scales?

$\longrightarrow$ Entanglement in many-body systems

# Definition of entanglement: Werner 1989

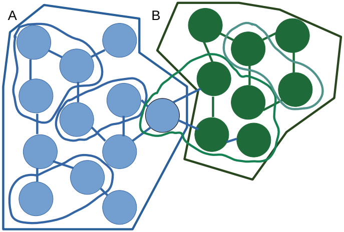



$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$$

- $p_i$ are (classical) probabilities
- $\rho_i^A$ and $\rho_i^B$ are the quantum states of the subsystems

$\rho$ is called **separable** $\rightarrow$ Non-separable states are called **entangled**

# Statement of the problem




Consider $N$ particles. Let us first divide it $A$ vs $B$

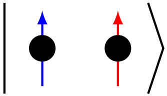
## Separable state

$$\rho = \sum_i p_i \left( |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B| \right)_i$$

Prove (from experimental data) that a state **cannot** be decomposed like this

# Simple two-spin example

Let us consider the following two-spin-$1/2$ state



It is a product (hence separable) state, and we have

$$\langle j_x \otimes j_x \rangle = \langle j_y \otimes j_y \rangle = 0$$
$$\langle j_z \otimes j_z \rangle = \frac{1}{4}$$

# Simple two-spin example

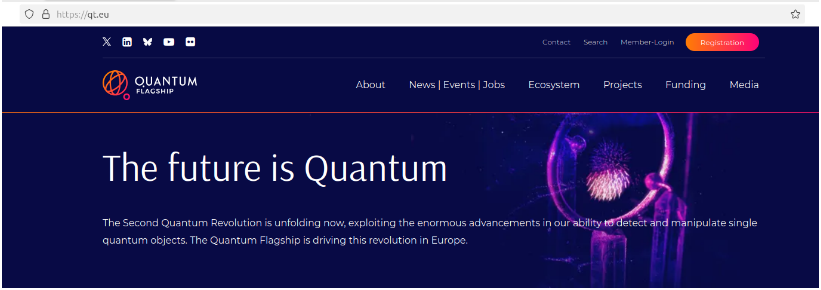
$$\frac{\left|\uparrow\ \downarrow\right\rangle - \left|\downarrow\ \uparrow\right\rangle}{\sqrt{2}}$$

For this state we have

$$\langle j_x \otimes j_x \rangle = \langle j_y \otimes j_y \rangle = \langle j_z \otimes j_z \rangle = -\frac{1}{4}$$

# Ok... but why?

- It is a way to witness that experiments have genuine quantum effects
- It is a necessary prerequisite for, e.g., quantum simulators, quantum sensors etc.
- Entanglement might have interesting connections with the physics of the many-body experiments
- It is an interesting (and very difficult) mathematical problem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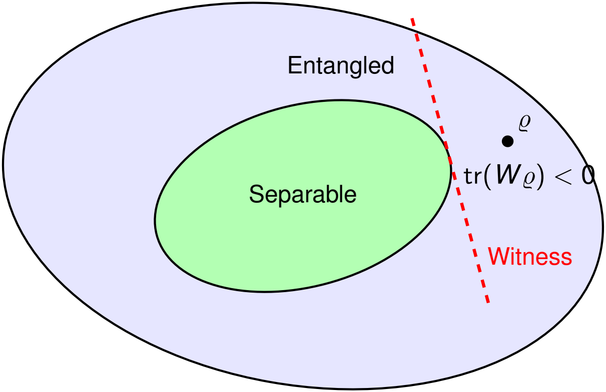
The future is Quantum
The Second Quantum Revolution is unfolding now, exploiting the enormous advancements in our ability to detect and manipulate single quantum objects. The Quantum Flagship is driving this revolution in Europe.

Note: In many-body experiments full state reconstruction is impossible

We look for methods which involve as simple measurements as possible

An idea that works very well in practice makes use of just variances of simple ensemble measurements

# Entanglement witnesses



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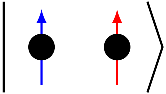

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# Simple two-spin example

$$\frac{\left| \uparrow \; \downarrow \right\rangle - \left| \downarrow \; \uparrow \right\rangle}{\sqrt{2}}$$

For this state we have

$$\langle j_x \otimes j_x \rangle = \langle j_y \otimes j_y \rangle = \langle j_z \otimes j_z \rangle = -\frac{1}{4}$$

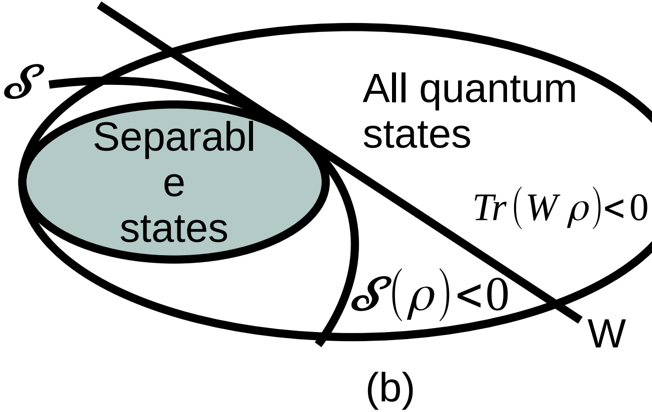
# Two-spin-1/2 entanglement witness

*All separable states* (i.e., $\varrho = \sum_i p_i(\varrho_A \otimes \varrho_B)$) must satisfy

$$\langle W \rangle := \langle j_x \otimes j_x \rangle + \langle j_y \otimes j_y \rangle + \langle j_z \otimes j_z \rangle \geq -1/4$$

The singlet instead has $\langle W \rangle = -3/4$

# Nonlinear entanglement witnesses



# Linear uncertainty relations

- Assume a single-party uncertainty relation

$$(\Delta j_x)^2 + (\Delta j_y)^2 \geq C_j$$

where $(\Delta j_x)^2 := \langle j_x^2 \rangle - \langle j_x \rangle^2$

- Take *collective observables*

$$J_x = j_x^A \otimes \mathbb{1}^B + \mathbb{1}^A \otimes j_x^B \qquad J_y = j_y^A \otimes \mathbb{1}^B + \mathbb{1}^A \otimes j_y^B$$

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- We obtain that

$$(\Delta J_x)^2 + (\Delta J_y)^2 < 2C_j \quad \Rightarrow \text{entanglement}$$

[H. F. Hofmann and S. Takeuchi, PRA, **68** 032103, (2003)] see also [Q. Y. He et al PRA 84, 022107 (2011); L. Dammeier, R. Schwonnek, R. F. Werner, New J. Phys. **17**, 093046 (2015)]

# Linear uncertainty relations

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- We obtain that

$$(\Delta J_x)^2 + (\Delta J_y)^2 < 2C_j \quad \Rightarrow \text{entanglement}$$

i.e., $(\Delta J_x)^2 + (\Delta J_y)^2 \geq 2C_j$ holds for all separable states

[H. F. Hofmann and S. Takeuchi, PRA, **68** 032103, (2003)] see also [Q. Y. He et al PRA 84, 022107 (2011); L. Dammeier, R. Schwonnek, R. F. Werner, New J. Phys. **17**, 093046 (2015)]

Note that $\langle J_k^2\rangle = \langle (j_k^A)^2\rangle + \langle (j_k^B)^2\rangle + 2\langle j_k^A \otimes j_k^B\rangle = \frac{1}{4} + \frac{1}{4} + 2\langle j_k^A \otimes j_k^B\rangle$.

Thus we have

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 = \frac{3}{2} + \langle W\rangle - \langle J_x\rangle^2 - \langle J_y\rangle^2 - \langle J_z\rangle^2$$

and in particular

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq 1$$

is and entanglement criterion which is strictly better than $\langle W\rangle \geq -1/4$

## *Proof.*

Separable states are defined as

$$\rho_{\mathrm{sep}} := \sum_k p_k \rho_k^A \otimes \rho_k^B$$

We have that

$$(\Delta J_x)^2_{\mathrm{sep}} + (\Delta J_y)^2_{\mathrm{sep}} \geq$$
$$\sum_k p_k \left[ (\Delta J_x)^2_{\rho_k^A \otimes \rho_k^B} + (\Delta J_y)^2_{\rho_k^A \otimes \rho_k^B} \right] =$$
$$\sum_k p_k \left[ (\Delta j_x)^2_{\rho_k^A} + (\Delta j_x)^2_{\rho_k^B} + (\Delta j_y)^2_{\rho_k^A} + (\Delta j_y)^2_{\rho_k^B} \right] \geq$$
$$\sum_k p_k (C_j + C_j) = 2C_j$$

# Generalization to many-body states

## Fully separable state

$$\rho = \sum_i p_i |\psi_1\rangle\langle\psi_1|_i \otimes \cdots \otimes |\psi_N\rangle\langle\psi_N|_i,$$

Take

$$J_k = j_k^{(1)} \otimes \mathbb{1}^{(2)} \otimes \cdots \otimes \mathbb{1}^{(N)} + \mathbb{1}^{(1)} \otimes j_k^{(2)} \otimes \cdots \otimes \mathbb{1}^{(N)} + \dots$$

(shorthand: $J_k = \sum_{n=1}^{N} j_k^{(n)}$)

- Given a single-particle uncertainty relation

$$(\Delta j_x)^2 + (\Delta j_y)^2 \geq C_j$$

- one can prove that

$$(\Delta J_x)^2 + (\Delta J_y)^2 < NC_j \quad \Rightarrow \text{entanglement}$$

# Entanglement of spin squeezed states

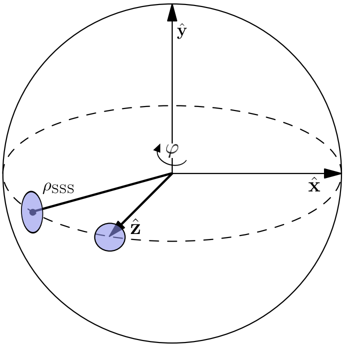
From $(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z\rangle|^2$ we define a spin-coherent state as

$$(\Delta J_x)^2 = (\Delta J_y)^2 = \frac{1}{2}|\langle J_z\rangle| = \frac{N}{4}$$

and *spin-squeezed states* as

$$|\langle J_z\rangle| \simeq \frac{N}{2} \; ; \qquad (\Delta J_x)^2 < \frac{N}{4}$$

# Entanglement of spin-squeezed states




Spin squeezing can be quantified with

$$N(\Delta J_x)^2 - \langle J_y \rangle^2 - \langle J_z \rangle^2 < 0 \quad \text{(which } \Rightarrow \text{ entanglement)}$$

They are also very useful for metrology

[A. Sørensen, L.M. Duan, J.I. Cirac, and P. Zoller, Nature **409**, 63 (2001);M. Kitagawa and M. Ueda, Phys. Rev. A **47**, 5138 (1993); D.J. Wineland, J. J. Bollinger, and W. M. Itano, Phys. Rev. A **50**, 67 (1994).]

# Generalized spin squeezing

From the uncertainty relation (for spin-$1/2$ particles)

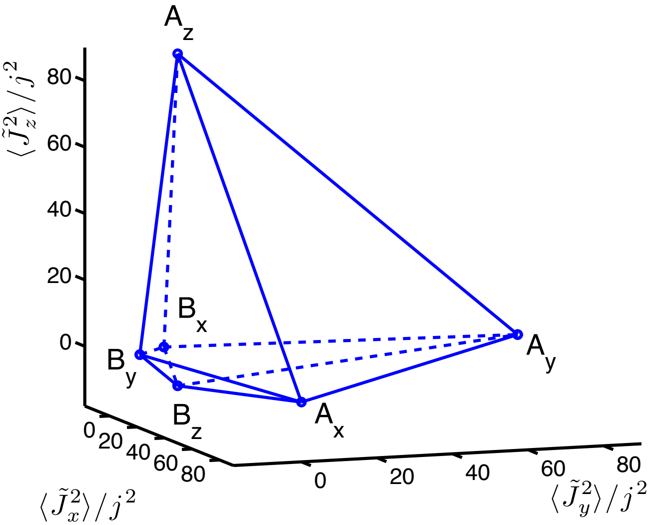
$$(\Delta j_x)^2 + (\Delta j_y)^2 + (\Delta j_z)^2 \geq \frac{1}{2}$$

it follows the full set of entanglement criteria

$$\begin{aligned}
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq \frac{N(N+2)}{4} \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq \frac{N}{2} \\
(N-1)\left[(\Delta J_x)^2 + (\Delta J_y)^2\right] - \langle J_z^2 \rangle &\geq \frac{N(N-2)}{4} \\
(N-1)\left[(\Delta J_x)^2\right] - \langle J_y^2 \rangle - \langle J_z^2 \rangle &\geq -\frac{N}{2}
\end{aligned}$$

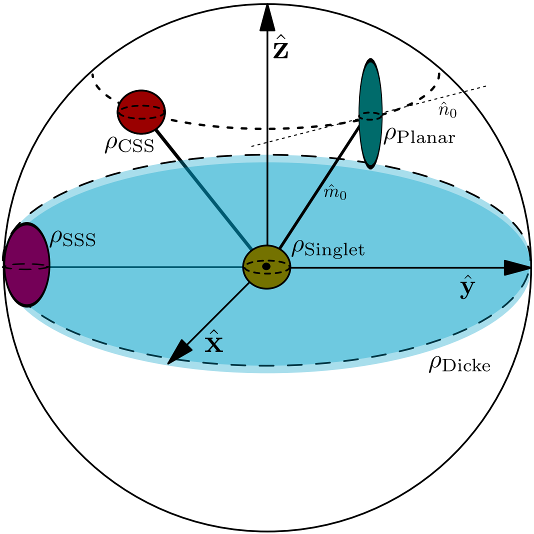
Violation of one of them implies entanglement.

[G. Tóth, C. Knapp, O. Gühne and H.J. Briegel, PRL **99**, 250405 (2007); PRA **79** 042334 (2009)]



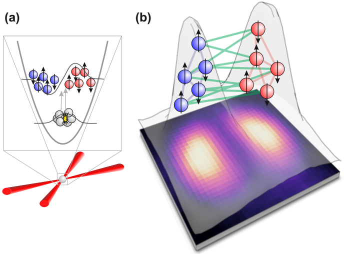
A_z
B_x
B_y
B_z
A_x
A_y
80
60
40
20
0
0
20
40
60
80
0
20
40
60
80
$\langle \tilde{J}_z^2 \rangle / j^2$
$\langle \tilde{J}_x^2 \rangle / j^2$
$\langle \tilde{J}_y^2 \rangle / j^2$

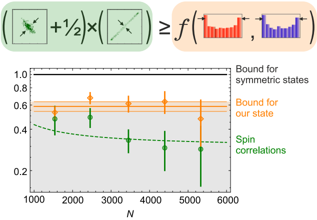
# Generalized Spin Squeezing: summary




[see e.g., G. Toth and I. Apellaniz JPA, 47(42):424006, (2014)]

# Experimental detection of entanglement between spatially separated modes

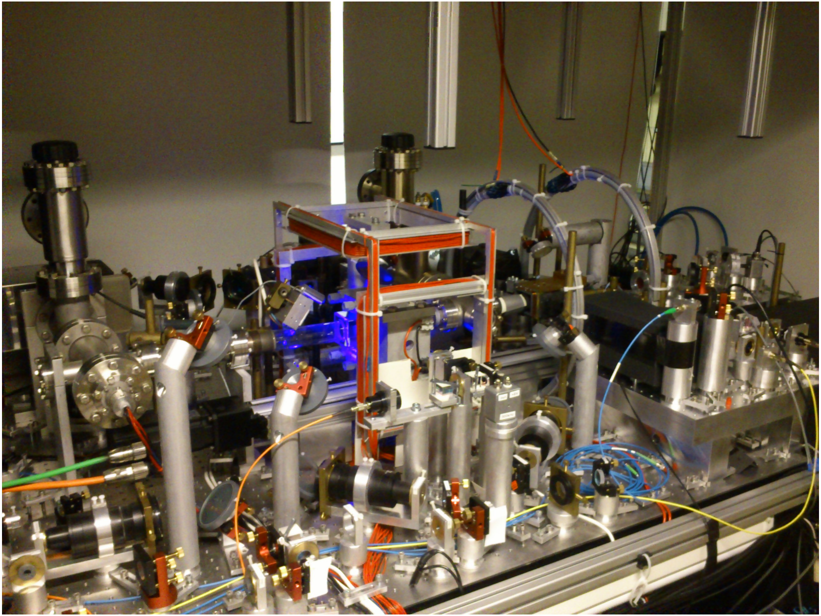



$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta J_x^-)^2 + (\Delta J_y^-)^2\right] \geq \frac{\left\langle J_x^2 + J_y^2 \right\rangle^2}{N(N+2)}$$

[see K. Lange, J. Peise, B. Lücke, I. Kruse, GV, I. Apellaniz, M. Kleinmann, G. Toth, C. Klempt, Science 360 416-418 (2018); see also P. Kunkel, M. Prüfer, H. Strobel, D. Linnemann, A. Frölian, T. Gasenzer, M. Gärttner, and M. K. Oberthaler, Science 360 413–416 (2018); M. Fadel, T. Zibold, B. Décamps, and P. Treutlein, Science 360 409–413 (2018)]

[see also GV, M. Fadel, I. Apellaniz, M. Kleinmann, B. Lücke, C. Klempt, G. Tóth Quantum 7, 914 (2023)]



# Further generalizations

1. systems of particles with spin $j > 1/2$

   *[GV et al. PRL 107, 240502 (2011); PRA 89, 032307 (2014)]*

2. considering $su(d)$ operators with $d = 2j + 1 > 2$

   *[GV et al. PRL 107, 240502 (2011); + in preparation]*

3. All possible linear uncertainty relations into a Covariance Matrix Criterion

   *[Gühne et al. PRL 99 130504 (2007); Gittsovich et al. PRA 78, 052319 (2008);PRA, 82 032306 (2010); PRA 81, 032333 (2010)]*

4. Even more general approaches based on moment matrix

   *[Bohnet-Waldraff, Braun, Giraud Phys. Rev. A 96, 032312 (2017); Frérot, Baccari, Acín PRX Quantum, 3(1), 010342 (2022)]*

+ many other related works, e.g., in continuous variables and related to metrology and Fisher information

# Criterion with the QFI

Similar criteria can be derived with the *Quantum Fisher Information*

$$F_Q(O,\varrho) := 2\sum_{kl} \frac{(\lambda_k-\lambda_l)^2}{\lambda_k+\lambda_l} |\langle k|O|l\rangle|^2 \quad \text{with} \quad \varrho = \sum_k \lambda_k |k\rangle\langle k| \qquad (1)$$

- For *pure states*: $F_Q(O,|\psi\rangle) = 4(\Delta O)^2_\psi$
- For *mixed states* $F_Q(O,|\psi\rangle) \leq 4(\Delta O)^2_\psi$

Thus, for separable states we have

$$F_Q(O,\varrho) \leq 4N\kappa_O \quad \text{with} \quad \kappa_O = \max_\psi (\Delta o)^2_\psi \qquad (2)$$

(the bound is on single-particle states)

[G. Tóth, I. Apellaniz, J. Phys. A: Math. Theo. 47 424006 (2014)]

# Entanglement detection with susceptibilities

At **thermal equilibrium**, variances are connected to *dynamic susceptibilities*:

$$(\Delta O)^2_\varrho = \int_0^\infty \mathrm{d}\omega \left(-\tfrac{1}{\pi}\mathrm{Im}\chi(\omega)\right) \coth\left(\omega/2T\right), \tag{3}$$

where

$$\chi(\omega) = \int \mathrm{d}t\; e^{i\omega t}\chi(t) \tag{4}$$

$$\chi(t) = -i\theta(t)\langle[O(t) - \langle O(t)\rangle, O^\dagger - \langle O^\dagger\rangle]\rangle \tag{5}$$

Similar relation holds for QFI with $\coth \to \tanh$

(due to Fluctuation-Dissipation theorems)

[P. Hauke, M. Heyl, L. Tagliacozzo, P. Zoller, Nat. Phys. 12, pages 778?782 (2016)]

# Entanglement detection from neutron scattering/structure factors

Collective variances are related to *average* two-particle correlations:

$$(\Delta O)^2_{\varrho} = \sum_n (\Delta o^{(n)})^2 + \sum_{n \neq m} \left( \langle o^{(n)} \otimes o^{(m)} \rangle - \langle o^{(n)} \rangle \langle o^{(m)} \rangle \right) \qquad (6)$$

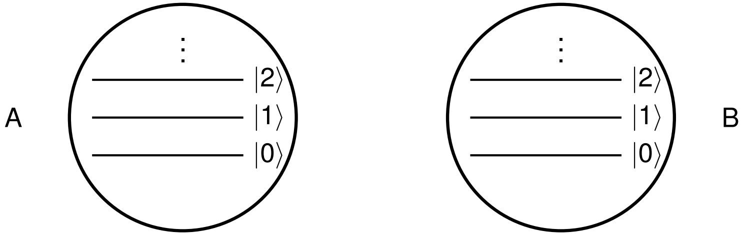
Those can be also extracted from scattering cross-sections:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\Omega \mathrm{d}\omega} \propto \frac{k_{\mathrm{in}}}{k_{\mathrm{out}}} \sum_{kl} \sum_{nm} c_{kl}^{(nm)} e^{iq(n-m)} \int e^{-i\omega t} \langle j_k^{(n)} \otimes j_k^{(m)}(t) \rangle \, \mathrm{d}t$$

Thus, these quantities can be also used for entanglement detection with similar methods

[O. Marty, *et al.* Phys. Rev. B 89, 125117 (2014); O. Marty, M. Cramer, G. Vitagliano, G. Tóth, M. Plenio arXiv:1708.06986 (2017)]

# Schmidt coefficients and entanglement monotones

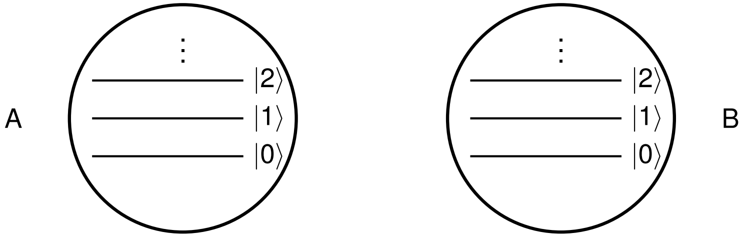



A bipartite *pure* state can be decomposed as:

$$|\Psi_{AB}\rangle = \sum_k \sqrt{\lambda_k}|k_A\rangle|k_B\rangle$$

with $\lambda_k \geq 0$ and $\sum_k \lambda_k = 1$.

# Schmidt coefficients and entanglement monotones

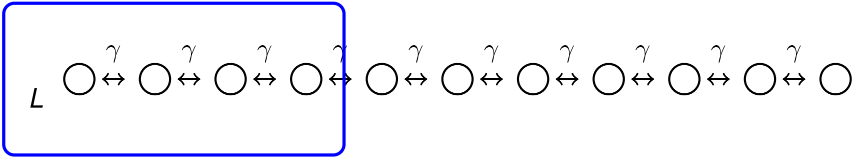



$$|\Psi_{AB}\rangle = \sum_k \sqrt{\lambda_k}|k_A\rangle|k_B\rangle \quad \Rightarrow \quad \varrho_A = \sum_k \lambda_k |k_A\rangle\langle k_A|$$

Entanglement monotones are functions of the $\lambda_k$, e.g.,

$$S(\Psi_{AB}) := -\operatorname{tr}(\varrho_A \log(\varrho_A)) = -\sum_k \lambda_k \log(\lambda_k)$$

# Area laws



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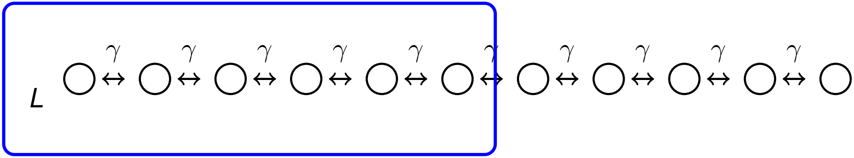


Scaling of entanglement with $L$ usually follows an **area law**:

$$S(L) \to \text{const.}$$

[L. Amico, R. Fazio, A. Osterloh, V. Vedral, Rev. Mod. Phys., 80(2) (2008); J. Eisert, M. Cramer, M.B. Plenio, Rev. Mod. Phys. 82, 277 (2010); N. Laflorencie, Phys. Rep. 646 (2016)]

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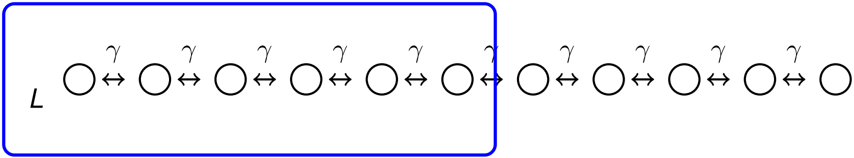


Such an area law is violated for **critical systems**:

$$S(L) \sim c \log(L)$$

[L. Amico, R. Fazio, A. Osterloh, V. Vedral, Rev. Mod. Phys., 80(2) (2008); J. Eisert, M. Cramer, M.B. Plenio, Rev. Mod. Phys. 82, 277 (2010); N. Laflorencie, Phys. Rep. 646 (2016)]

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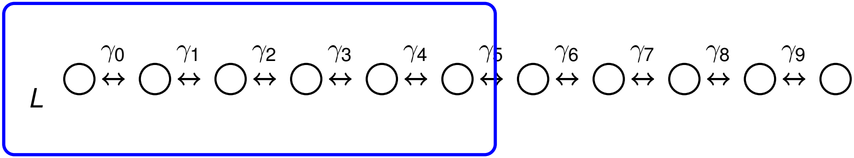
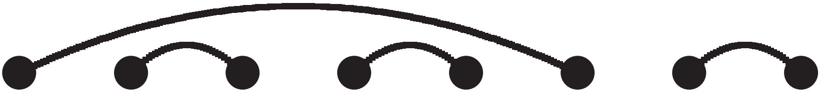


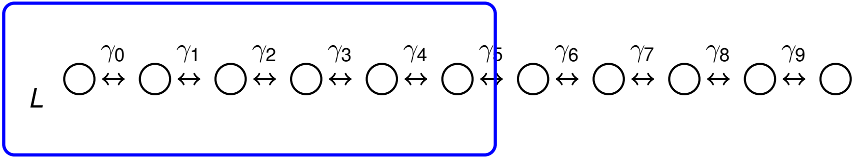



For *random* disorder it is:

$$S(L) \sim c \log 2 \log(L)$$

[L. Amico, R. Fazio, A. Osterloh, V. Vedral, Rev. Mod. Phys., 80(2) (2008); J. Eisert, M. Cramer, M.B. Plenio, Rev. Mod. Phys. 82, 277 (2010); N. Laflorencie, Phys. Rep. 646 (2016)]

# Area laws

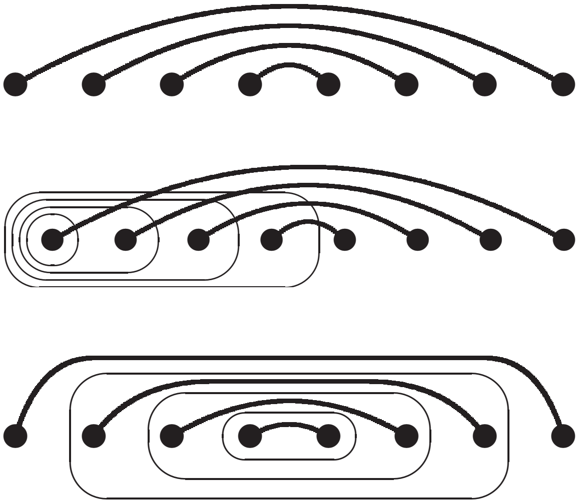



For *engineered* disorder it is maximally violated:

$$S(L) = L$$

[GV, A. Riera, J.I. Latorre, New J. Phys. 12, 113049 (2010)]

# Area laws



[GV, A. Riera, J.I. Latorre New J. Phys. 12 113049 (2010)]

# Entanglement monotones

Entanglement monotones are defined as

$$\sum_k p_k \mathcal{E}(A_k \rho A_k^\dagger / p_k) \leq \mathcal{E}(\rho),$$

the map $A_k \rho A_k^\dagger$ is a *Local Operation and Classical Communication* (LOCC)

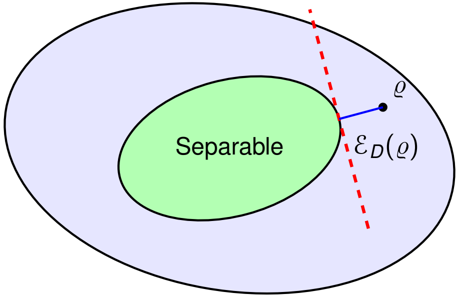
(bipartite) examples: Schmidt rank, Concurrence, Entanglement of Formation...

Multipartite extension have to consider **all possible bipartitions**

[S. Liu, M. Fadel, Q. He, M. Huber, GV Quantum 8, 1236 (2024); S Liu, Q He, M Huber, GV arXiv:2405.03261 ]

# Distance-based class of monotones

$$\mathcal{E}_D(\varrho) = \min_{\sigma \in \mathrm{SEP}} D(\varrho, \sigma), \tag{7}$$




One example such distance is called

$$\mathcal{E}_{BSA}(\varrho) = \min t \in [0,1] : \quad \varrho = (1-t)\,\sigma + t\,\delta\varrho \ ,$$

*(Best Separable Approximation)*

[Lewenstein, Sanpera PRL 80, 2261 (1998) and Karnas, Lewenstein, J. Phys. A 34, 6919 (2001)]

These can be bounded from entanglement witnesses

[F. G. S. L. Brandão, Phys. Rev. A 72, 022310 (2005); M. Cramer, M. B. Plenio, and H. Wunderlich Phys. Rev. Lett. 106, 020401 (2011); M Cramer *et al.*, Nat. Comm. 4, 2161 (2013)]

# Lower-bounding monotones from variance-based criteria

Take

$$\mathcal{S}(\rho) := \sum_k (\Delta O_k)^2 - \langle B \rangle \geq 0,$$

$$\lambda_{\max}(B) = -n$$

One can prove that

$$\mathcal{BSA}(\rho) \geq -\mathcal{S}(\rho)/n$$

[M. Fadel, A. Usui, M. Huber, N. Friis, GV PRL 127, 010401 (2021)]

# Upper-bounding monotones from ansatz separable states

An upper bound can be found as:

$$\mathcal{E}_{BSA}(\varrho) \geq \min_{t \in [0,1]} \operatorname{tr}(\varrho - (1-t)\sigma), \tag{8}$$

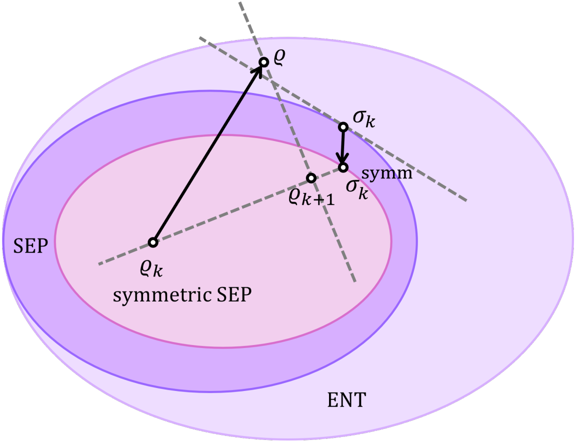
for **any** $\sigma \in \mathrm{SEP}$

Iteratively we can

1. Consider $\sigma_{K-1}$ from previous iteration (with e.g., $\sigma_0 = \mathbb{1}/2^N$).
2. Choose a new $|\psi_1 \ldots \psi_N\rangle_K$ maximizing the "overlap" with $\varrho - \sigma_{K-1}$.
3. Add it to the ensemble $\{p_k, |\phi_k\rangle\}$.
4. Find new probabilities $\{p_k\}$ by minimizing $D(\varrho, \sigma_K)$.

Symmetries can be taken into account

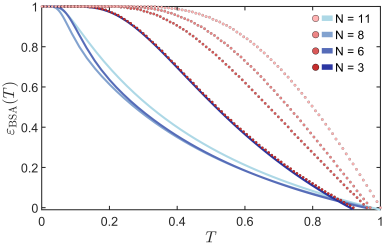
# Upper-bounding monotones from ansatz separable states

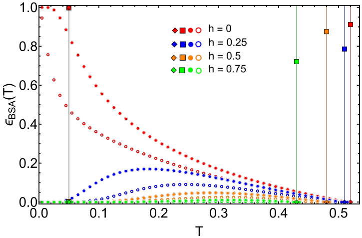



[J. Maté, A. Usui, O. Gühne, GV arXiv:2504.07814 (2025)]

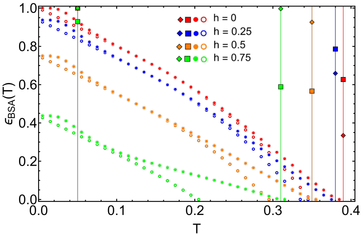
# Bounding BSA in equilibrium states

$$H = \frac{g}{N}\left(J_x^2 + J_y^2\right) + \frac{g_z}{N}J_z^2 + hJ_z. \quad (9)$$



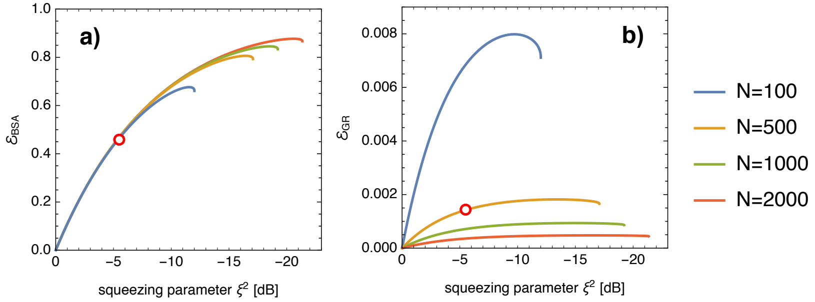




Ferromagnetic case



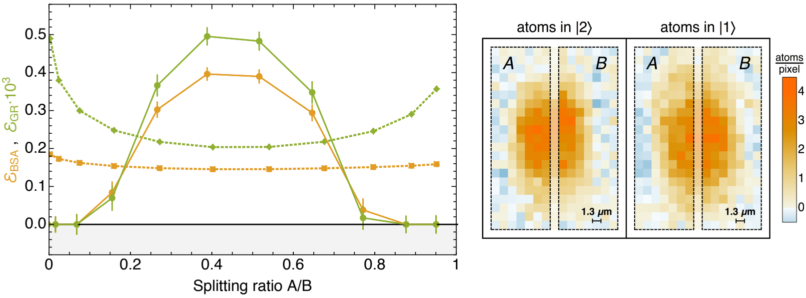

Anti-Ferromagnetic case

# Bounding BSA in cold-atom experiments




$$\xi^2 := N(\Delta J_z)^2/\langle J_x\rangle^2 \to \mathcal{BSA}(\rho) \geq \langle J_x\rangle/(N/2)\left(1 - \sqrt{\xi^2}\right)$$

[M. Fadel, A. Usui, M. Huber, N. Friis, GV PRL 127, 010401 (2021)]
[Data from R. Schmied, J.-D. Bancal, B. Allard, M. Fadel, V. Scarani, P. Treutlein, and N. Sangouard, Science 352, 441 (2016)]




[M. Fadel, A. Usui, M. Huber, N. Friis, GV PRL 127, 010401 (2021)]
[Data from M. Fadel, T. Zibold, B. Décamps, and P. Treutlein, Science 360 409–413 (2018)]

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